Effective transmission conditions for Hamilton-Jacobi equations defined on two domains separated by an oscillatory interface

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Abstract

We consider a family of optimal control problems in the plane with dynamics and running costs possibly discontinuous across an oscillatory interface Γ_ε. The oscillations of the interface have small period and amplitude, both of the order of ε, and the interfaces Γ_ε tend to a straight line Γ. We study the asymptotic behavior as $\varepsilon \to 0$. We prove that the value function tends to the solution of Hamilton-Jacobi equations in the two half-planes limited by Γ, with an effective transmission condition on Γ keeping track of the oscillations of Γ_ε.

1 Introduction

The goal of this paper is to study the asymptotic behavior as $\varepsilon \to 0$ of the value function of an optimal control problem in $\mathbb{R}^2$ in which the running cost and dynamics may jump across a periodic oscillatory interface Γ_ε, when the oscillations of Γ_ε have a small amplitude and period, both of the order of ε. The interface Γ_ε separates two unbounded regions of $\mathbb{R}^2$, Ω_ε^L and Ω_ε^R. To characterize the optimal control problem, one has to specify the admissible dynamics at a point $x \in \Gamma_\varepsilon$: in our setting, no mixture is allowed at the interface, i.e. the admissible dynamics are the ones corresponding to the subdomain Ω_ε^L **and** entering Ω_ε^L, or corresponding to the subdomain Ω_ε^R **and** entering Ω_ε^R. Hence the situation differs from those studied in the articles of G. Barles, A. Briani and E. Chasseigne [5, 6] and of G. Barles, A. Briani, E. Chasseigne and N. Tchou [7], in which mixing is allowed at the interface. The optimal control problem under consideration has been first studied in [16]: the value function is characterized as the viscosity solution of a Hamilton-Jacobi equation with special transmission conditions on Γ_ε; a comparison principle for this problem is proved in [16] with arguments from the theory of optimal control similar to those introduced in [5, 6]. In parallel to [16], Imbert and Monneau have studied similar problems from the viewpoint of PDEs, see [12], and have obtained comparison results for quasi-convex Hamiltonians. In particular, [12] contains a characterization of the viscosity solution of the transmission problem with a reduced set of test-functions; this characterization will be used in the present work. Note that [16, 12] can be seen as extensions of articles devoted to the analysis of Hamilton-Jacobi equations on networks, see [1, 13, 2, 11], because the notion of interface used there can be seen as a generalization of the notion of vertex (or junction) for a network.

We will see that as ε tends to 0, the value function converges to the solution of an effective

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problem related to a flat interface Γ, with Hamilton-Jacobi equations in the half-planes limited by Γ and a transmission condition on Γ.
Whereas the partial differential equation far from the interface is unchanged, the main difficulty consists in finding the effective transmission condition on Γ. Naturally, the latter depends on the dynamics and running cost but also keeps memory of the vanishing oscillations. The present work is closely related to two recent articles, [3] and [10], about singularly perturbed problems leading to effective Hamilton-Jacobi equations on networks. Indeed, an effective Hamiltonian corresponding to trajectories staying close to the junction was first obtained in [3] as the limit of a sequence of ergodic constants corresponding to larger and larger bounded subdomains. This construction was then used in [10] in a different case. Let us briefly describe the singular perturbation problems studied in [3] and [10]: in [3], some of the authors of the present paper study a family of star-shaped planar domains D^ε made of N non intersecting semi-infinite strips of thickness ε and of a central region whose diameter is proportional to ε. As $\varepsilon \to 0$, the domains D^ε tend to a network $\mathcal{G}$ made of N half-lines sharing an endpoint O, named the vertex or junction point. For infinite horizon optimal control problems in which the state is constrained to remain in the closure of D^ε, the value function tends to the solution of a Hamilton-Jacobi equation on $\mathcal{G}$, with an effective transmission condition at O. In [10], Galise, Imbert and Monneau study a family of Hamilton-Jacobi equations in a simple network composed of two half-lines with a perturbation of the Hamiltonian localized in a small region close to the junction.
In the proof of convergence, we will see that the main technical point lies in the construction of correctors and in their use in the perturbed test-function method of Evans, see [8]. As in [3] and [10], an important difficulty comes from the unboundedness of the domain in which the correctors are defined. The strategies for passing to the limit in [3] and [10] differ: the method proposed in [3] consists of contructing an infinite family of correctors related to the vertex, while in [10], only one corrector related to the vertex is needed thanks to the use of the above mentioned reduced set of test-functions. Arguably, the strategy proposed in [3] is more natural and that in [10] is simpler. For this reason, the technique implemented in the present work for proving the convergence to the effective problem will be closer to the one proposed in [10]. Note that similar techniques are used in the very recent work [9], which deals with applications to traffic flows. The question of the correctors in unbounded domains has recently been addressed by P-L. Lions in his lectures at Collège de France, [14], precisely in january and february 2014: the lectures dealt with recent and still unpublished results obtained in collaboration with T. Souganidis on the asymptotic behavior of solutions of Hamilton-Jacobi equations in a periodic setting with some localized defects. Finally, we stress the fact that the technique proposed in the present work is not specific to the transmission condition imposed on Γ_ε.

The paper is organized as follows: in the remaining part of § 1, we set the problem and give the main result. In Section 2, we show that the problem is equivalent to a more convenient one, set in a straightened fixed geometry. In § 3, we study the asymptotic behavior far from the interface and introduce some ingredients that will be useful to define the effective transmission condition. In § 4, we define the effective cost/Hamiltonian for moving along the effective interface, and related correctors. This is of course a key step in the study of the asymptotic behavior. Section 5 deals with further properties of the correctors, in particular their growth at infinity. The comparison result for the effective problem is stated in § 6, and the proof of the main convergence theorem is written in § 7.

1.1 The geometry

Let (e_1, e_2) be an orthonormal basis of $\mathbb{R}^2$: $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$-function, periodic with period 1. For $\varepsilon > 0$, let $(\Omega_\varepsilon^L, \Gamma_\varepsilon, \Omega_\varepsilon^R)$ be the following partition of $\mathbb{R}^2$:

$$\Gamma_\varepsilon = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \varepsilon g\left(\frac{x_2}{\varepsilon}\right)\right\}, \tag{1.1}$$

$$\Omega_\varepsilon^L = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < \varepsilon g\left(\frac{x_2}{\varepsilon}\right)\right\}, \qquad \Omega_\varepsilon^R = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > \varepsilon g\left(\frac{x_2}{\varepsilon}\right)\right\}. \tag{1.2}$$

Note that $\partial\Omega_\varepsilon^L = \partial\Omega_\varepsilon^R = \Gamma_\varepsilon$. For $x \in \Gamma_\varepsilon$, the vector

$$n_\varepsilon(x) = \begin{pmatrix} 1 \\ -g'(\frac{x_2}{\varepsilon}) \end{pmatrix} \tag{1.3}$$

is normal to Γ_ε and oriented from Ω_ε^L to Ω_ε^R. With

$$\sigma^L = -1, \qquad \sigma^R = 1, \tag{1.4}$$

the vector $\sigma^i n_\varepsilon(x)$ is normal to Γ_ε at the point $x \in \Gamma_\varepsilon$ and points toward Ω_ε^i, for $i = L, R$.
The geometry obtained at the limit when $\varepsilon \to 0$ can also be found by taking $g = 0$ in the definitions above: let $(\Omega^L, \Gamma, \Omega^R)$ be the partition of $\mathbb{R}^2$ defined by

$$\Gamma = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\right\}, \tag{1.5}$$

$$\Omega^L = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0)\right\}, \quad \Omega^R = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0)\right\}. \tag{1.6}$$

One sees that $\partial\Omega^L = \partial\Omega^R = \Gamma$ and that for all $x \in \Gamma$, the unit normal vector to Γ at x pointing toward Ω^R is $n(x) = e_1$. The two kinds of geometry are represented in Figure 1.

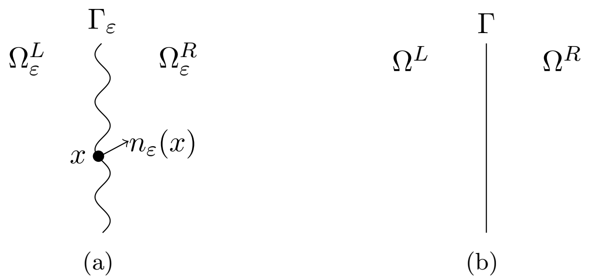

Figure 1: (a): Γ_ε is an oscillating interface with an amplitude and period of ε . (b): the geometry obtained at the limit when $\varepsilon \to 0$

1.2 The optimal control problem in $\Omega_\varepsilon^L \cup \Omega_\varepsilon^R \cup \Gamma_\varepsilon$

We consider infinite-horizon optimal control problems which have different dynamics and running costs in the regions Ω_ε^i, $i = L, R$. The sets of controls associated to the index $i = L, R$ will be called A^i; similarly, the notations f^i and ℓ^i will be used for the dynamics and running costs. The following assumptions will be made in all the work

1.2.1 Standing Assumptions

[H0] A is a metric space (one can take $A = \mathbb{R}^m$). For $i = L, R$, A^i is a non empty compact subset of A and $f^i : \mathbb{R}^2 \times A^i \to \mathbb{R}^2$ is a continuous bounded function. The sets A^i are disjoint. Moreover, there exists $L_f > 0$ such that for any $i = L, R$, $x, y \in \mathbb{R}^2$ and $a \in A^i$,

$$|f^i(x,a) - f^i(y,a)| \le L_f |x - y|.$$

Define $M_f = \max_{i=L,R} \sup_{x\in\mathbb{R}^2, a\in A^i} |f^i(x,a)|$. The notation $F^i(x)$ will be used for the set $F^i(x) = \{f^i(x,a), a \in A^i\}$.

[H1] For $i = L, R$, the function $\ell^i : \mathbb{R}^2 \times A^i \to \mathbb{R}$ is continuous and bounded. There is a modulus of continuity ω_ℓ such that for any $i = L, R$, $x, y \in \mathbb{R}^2$ and $a \in A^i$,

$$|\ell^i(x,a) - \ell^i(y,a)| \le \omega_\ell(|x - y|).$$

Define $M_\ell = \max_{i=L,R} \sup_{x\in\mathbb{R}^2, a\in A^i} |\ell^i(x,a)|$.

[H2] For any $i = L, R$ and $x \in \mathbb{R}^2$, the non empty set $\mathrm{FL}^i(x) = \{(f^i(x,a), \ell^i(x,a)), a \in A^i\}$ is closed and convex.

[H3] There is a real number $\delta_0 > 0$ such that for $i = L, R$ and all $x \in \Gamma_\varepsilon$, $B(0, \delta_0) \subset F^i(x)$.

We stress the fact that all the results below hold provided the latter assumptions are satisfied, although, in order to avoid tedious repetitions, we will not mention them explicitly in the statements.
We refer to [2] and [16] for comments on the assumptions and the genericity of the model, stressing in particular that the sets A^L, A^R can always been supposed disjoint.

1.2.2 The optimal control problem

Let the closed set $\mathcal{M}_\varepsilon$ be defined as follows:

$$\mathcal{M}_\varepsilon = \left\{(x,a);\ x \in \mathbb{R}^2, \quad a \in A^i \text{ if } x \in \Omega^i_\varepsilon,\ i = L, R, \ \text{ and } a \in A^L \cup A^R \text{ if } x \in \Gamma_\varepsilon\right\}. \tag{1.7}$$

The dynamics f_ε is a function defined in $\mathcal{M}_\varepsilon$ with values in $\mathbb{R}^2$:

$$\forall (x,a) \in \mathcal{M}_\varepsilon, \qquad f_\varepsilon(x,a) = \begin{cases} f^i(x,a) & \text{if } x \in \Omega^i_\varepsilon, \\ f^i(x,a) & \text{if } x \in \Gamma_\varepsilon \text{ and } a \in A^i. \end{cases}$$

The function f_ε is continuous on $\mathcal{M}_\varepsilon$ because the sets A^i are disjoint. Similarly, let the running cost $\ell_\varepsilon : \mathcal{M}_\varepsilon \to \mathbb{R}$ be given by

$$\forall (x,a) \in \mathcal{M}_\varepsilon, \qquad \ell_\varepsilon(x,a) = \begin{cases} \ell^i(x,a) & \text{if } x \in \Omega^i_\varepsilon, \\ \ell^i(x,a) & \text{if } x \in \Gamma_\varepsilon \text{ and } a \in A^i. \end{cases}$$

For $x \in \mathbb{R}^2$, the set of admissible trajectories starting from x is

$$\mathcal{T}_{x,\varepsilon} = \left\{ \begin{array}{ll} (y_x, a) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}_\varepsilon) : & y_x \in \mathrm{Lip}(\mathbb{R}^+; \mathbb{R}^2), \\ & y_x(t) = x + \displaystyle\int_0^t f_\varepsilon(y_x(s), a(s))ds \quad \forall t \in \mathbb{R}^+ \end{array} \right\}. \tag{1.8}$$

The cost associated to the trajectory $(y_x, a) \in \mathcal{T}_{x,\varepsilon}$ is

$$\mathcal{J}_\varepsilon(x; (y_x, a)) = \int_0^\infty \ell_\varepsilon(y_x(t), a(t)) e^{-\lambda t} dt, \tag{1.9}$$

with $\lambda > 0$. The value function of the infinite horizon optimal control problem is

$$v_\varepsilon(x) = \inf_{(y_x,a)\in\mathcal{T}_{x,\varepsilon}} \mathcal{J}_\varepsilon(x;(y_x,a)). \tag{1.10}$$

Proposition 1.1. *The value function v_ε is bounded and continuous in $\mathbb{R}^2$.*

Proof. This result is classical and can be proved with the same arguments as in [4]. □

1.3 The Hamilton-Jacobi equation

Similar optimal control problems have recently been studied in [2, 11, 16, 12]. It turns out that v_ε can be characterized as the viscosity solution of a Hamilton-Jacobi equation with a discontinuous Hamiltonian, (once the notion of viscosity solution has been specially tailored to cope with the above mentioned discontinuity). We briefly recall the definitions used e.g. in [16].

1.3.1 Test-functions

Definition 1.2. *For $\varepsilon > 0$, the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an admissible (ε)-test-function if ϕ is continuous in $\mathbb{R}^2$ and for any $i \in \{L, R\}$, $\phi|_{\overline{\Omega^i_\varepsilon}} \in \mathcal{C}^1(\overline{\Omega^i_\varepsilon})$.*
The set of admissible test-functions is noted $\mathcal{R}_\varepsilon$. If $\phi \in \mathcal{R}_\varepsilon$, $x \in \Gamma_\varepsilon$ and $i \in \{L, R\}$, we set $D\phi^i(x) = \lim_{\substack{x'\to x\\ x'\in\Omega^i_\varepsilon}} D\phi(x')$.

1.3.2 Hamiltonians

For $i = L, R$, let the Hamiltonians $H^i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $H_{\Gamma_\varepsilon} : \Gamma_\varepsilon \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H^i(x,p) = \max_{a\in A^i}(-p\cdot f^i(x,a) - \ell^i(x,a)), \tag{1.11}$$

$$H_{\Gamma_\varepsilon}(x,p^L,p^R) = \max\{\, H^{+,L}_{\Gamma_\varepsilon}(x,p^L), H^{+,R}_{\Gamma_\varepsilon}(x,p^R)\}, \tag{1.12}$$

where, with $n_\varepsilon(x)$ and σ^i defined in § 1.1,

$$H^{+,i}_{\Gamma_\varepsilon}(x,p) = \max_{a\in A^i \text{ s.t. } \sigma^i f^i(x,a)\cdot n_\varepsilon(x)\geq 0}(-p\cdot f^i(x,a) - \ell^i(x,a)), \quad \forall x\in\Gamma_\varepsilon, \forall p\in\mathbb{R}^2. \tag{1.13}$$

1.3.3 Definition of viscosity solutions

We now recall the definition of a viscosity solution of

$$\lambda u + \mathcal{H}_\varepsilon(x, Du) = 0. \tag{1.14}$$

Definition 1.3.
- *An upper semi-continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a subsolution of (1.14) if for any $x \in \mathbb{R}^2$, any $\phi \in \mathcal{R}_\varepsilon$ s.t. $u - \phi$ has a local maximum point at x, then*

$$lu(x) + H^i(x, D\phi^i(x)) \leq 0, \qquad \textit{if } x\in\Omega^i_\varepsilon, \tag{1.15}$$

$$lu(x) + H_{\Gamma_\varepsilon}(x, D\phi^L(x), D\phi^R(x)) \leq 0, \qquad \textit{if } x\in\Gamma_\varepsilon, \tag{1.16}$$

see Definition § 1.1 for the meaning of $D\phi^i(x)$ if $x \in \Gamma_\varepsilon$.

- *A lower semi-continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a supersolution of (1.14) if for any $x \in \mathbb{R}^2$, any $\phi \in \mathcal{R}_\varepsilon$ s.t. $u - \phi$ has a local minimum point at x, then*

$$lu(x) + H^i(x, D\phi^i(x)) \geq 0, \qquad \textit{if } x\in\Omega^i_\varepsilon, \tag{1.17}$$

$$lu(x) + H_{\Gamma_\varepsilon}(x, D\phi^L(x), D\phi^R(x)) \geq 0 \qquad \textit{if } x\in\Gamma_\varepsilon. \tag{1.18}$$

- *A continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a viscosity solution of* (1.14) *if it is both a viscosity sub and supersolution of* (1.14)*.*

1.3.4 Characterization of v_ε as a viscosity solution of (1.14)

The following theorem will be proved below, see Theorem 2.7, by finding an equivalent optimal control problem in a straightened fixed geometry and using some results contained in [16]:

Theorem 1.4. *The value function v_ε defined in* (1.10) *is the unique bounded viscosity solution of* (1.14)*.*

1.4 Main result and organization of the paper

We now state our main result:

Theorem 1.5. *As $\varepsilon \to 0$, v_ε converges uniformly to v the unique bounded viscosity solution of*

$$\lambda v(z) + H^i(z, Dv(z)) = 0 \qquad \text{if } z \in \Omega^i, \tag{1.19}$$

$$\lambda v(z) + \max\left(E(z_2, \partial_{z_2} v(z)), H_\Gamma(z, Dv^L(z), Dv^R(z))\right) = 0 \qquad \text{if } z = (0, z_2) \in \Gamma, \tag{1.20}$$

which we note for short

$$\lambda v + \mathcal{H}(x, Dv) = 0. \tag{1.21}$$

The Hamiltonians H^i, H_Γ and E are respectively defined in (1.11)*,* (3.4) *below, and* (4.18) *below.*

Let us list the notions which are needed by Theorem 1.5 and give a few comments:

1. Problem (1.21) is a transmission problem across the interface Γ, with the effective transmission condition (1.20). The notion of viscosity solutions of (1.21) is similar to the one proposed in Definition 1.3, replacing Γ_ε with Γ.

2. Note that the Hamilton-Jacobi equations in Ω^L and Ω^R are directly inherited from (1.15): this is quite natural, since the interface Γ_ε oscillates with an amplitude of the order of ε, which therefore vanishes as $\varepsilon \to 0$.

3. The Hamiltonian H_Γ appearing in the effective transmission condition at the junction is defined in § 3.2, precisely in (3.4); it is built by considering only the dynamics related to Ω^i which point from Γ toward Ω^i, for $i = L, R$.

4. The effective Hamiltonian E is the only ingredient in the effective problem that keeps track of the oscillations of Γ_ε, i.e. of the function g. It is constructed in § 4, see (4.18), as the limit of a sequence of ergodic constants related to larger and larger bounded subdomains. This is reminiscent of a construction first performed in [3] for singularly perturbed problems in optimal control leading to Hamilton-Jacobi equations posed on a network. A similar construction can also be found in [10].

5. For proving Theorem 1.5, the chosen strategy is reminiscent of [10], because it relies on the construction of a single corrector, whereas the method proposed in [3] requires the construction of an infinite family of correctors. This will be done in § 4 and the slopes at infinity of the correctors will be studied in § 5.

2 Straightening the geometry

It will be convenient to use a change of variables depending on ε and set the problem in a straightened and fixed geometry.

2.1 A change of variables

The following change of variables can be used to write the optimal control problem in a fixed geometry: for $x \in \mathbb{R}^2$, take $z = G(x) = \begin{pmatrix} x_1 - \varepsilon g(\frac{x_2}{\varepsilon}) \\ x_2 \end{pmatrix}$. We see that $G^{-1}(x) = \begin{pmatrix} x_1 + \varepsilon g(\frac{x_2}{\varepsilon}) \\ x_2 \end{pmatrix}$. The oscillatory interface Γ_ε is mapped onto $\Gamma = \{z : z_1 = 0\}$ by G. The Jacobian of G is

$$J_\varepsilon(x) = \begin{pmatrix} 1 & -g'(\frac{x_2}{\varepsilon}) \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

and it inverse is $J_\varepsilon^{-1}(x) = \begin{pmatrix} 1 & g'\left(\frac{x_2}{\varepsilon}\right) \\ 0 & 1 \end{pmatrix}$. The following properties will be useful: for any $x \in \mathbb{R}^2$,

$$J_\varepsilon(G^{-1}(x)) = J_\varepsilon(x) \qquad \text{and} \qquad J_\varepsilon^{-1}(G^{-1}(x)) = J_\varepsilon^{-1}(x), \tag{2.2}$$

$$\sup_{X\in\mathbb{R}^2,|X|\le 1} |J_\varepsilon(x)X| \le \sqrt{2}(1+ \| g' \|_\infty) \qquad \text{and} \qquad \sup_{X\in\mathbb{R}^2,|X|\le 1} |J_\varepsilon^{-1}(x)X| \le \sqrt{2}(1+ \| g' \|_\infty), \tag{2.3}$$

where $|\cdot|$ stands for the euclidean norm. Note that (2.2) holds because G and G^{-1} leave x_2 unchanged, and J_ε only depends on x_2.

2.2 The optimal control problem in the straightened geometry

For $i = L, R$, we define the new dynamics $\tilde f_\varepsilon^i$ and running costs $\tilde\ell_\varepsilon^i$ as

$$\begin{aligned} \tilde f_\varepsilon^i : \quad \bar\Omega^i \times A^i &\to \mathbb{R}^2, \qquad (z,a) \mapsto J_\varepsilon(z) f^i\left(G^{-1}(z), a\right), &(2.4)\\ \tilde\ell_\varepsilon^i : \quad \bar\Omega^i \times A^i &\to \mathbb{R}, \qquad (z,a) \mapsto \ell^i\left(G^{-1}(z), a\right). &(2.5)\end{aligned}$$

We deduce the following properties from the standing assumptions [H0]-[H3]:

$[\widetilde{\mathrm{H}}0]_\varepsilon$ For $i = L, R$ and $\varepsilon > 0$, the function $\tilde f_\varepsilon^i$ is continuous and bounded. Moreover, there exists $\tilde L_f(\varepsilon) > 0$ and $\tilde M_f > 0$ such that for any $z, z' \in \bar\Omega^i$ and $a \in A^i$,

$$|\tilde f_\varepsilon^i(z,a) - \tilde f_\varepsilon^i(z',a)| \le \tilde L_f(\varepsilon)|z - z'|,$$
$$|\tilde f_\varepsilon^i(z,a)| \le \tilde M_f.$$

$[\widetilde{\mathrm{H}}1]_\varepsilon$ For $i = L, R$ and $\varepsilon > 0$, the function $\tilde\ell_\varepsilon^i$ is continuous and bounded. Moreover, if we set $\tilde\omega_\ell(t) = \omega_\ell(\sqrt{2}(1+ \| g' \|_\infty)t)$, then for any $z, z' \in \bar\Omega^i$ and $a \in A^i$,

$$|\tilde\ell_\varepsilon^i(z,a) - \tilde\ell_\varepsilon^i(z',a)| \le \tilde\omega_\ell(|z - z'|),$$
$$|\tilde\ell_\varepsilon^i(x,a)| \le M_\ell,$$

the constants M_ℓ and the modulus of continuity $\omega_\ell(\cdot)$ being introduced in [H1].

$[\widetilde{\mathrm{H}}2]_\varepsilon$ For any $i = L, R$, $\varepsilon > 0$ and $x \in \bar\Omega^i$, the non empty set $\tilde{\mathrm{FL}}_\varepsilon^i(x) = \{(\tilde f_\varepsilon^i(x,a), \tilde\ell_\varepsilon^i(x,a)), a \in A^i\}$ is closed and convex.

$[\widetilde{\mathrm{H}}3]_\varepsilon$ For any $i = L, R$ and $\varepsilon > 0$, if we set $\tilde{\delta}_0 = \frac{\delta_0}{\sqrt{2}(1+\|g'\|_\infty)}$, then for any $z \in \Gamma$, $B(0, \tilde{\delta}_0) \subset \tilde{F}^i_\varepsilon(z) = \{\tilde{f}^i_\varepsilon(z,a), a \in A^i\}$.

Properties $[\widetilde{\mathrm{H}}0]_\varepsilon$ and $[\widetilde{\mathrm{H}}1]_\varepsilon$ result from direct calculations. Property $[\widetilde{\mathrm{H}}2]_\varepsilon$ comes from the fact that linear maps preserve the convexity property. In order to prove $[\widetilde{\mathrm{H}}3]_\varepsilon$, take $i = L, R$, $z = (0, z_2) \in \Gamma$ and $p \in B(0, \tilde{\delta}_0)$. We look for $a \in A^i$ such that $\tilde{f}^i_\varepsilon(z,a) = p$. Using (2.3), we see that $|J^{-1}_\varepsilon(z)p| \le \sqrt{2}(1 + \| g' \|_\infty)\tilde{\delta}_0 \le \delta_0$. Thus from [H3], since $G^{-1}(z) \in \Gamma_\varepsilon$, there exists $\bar{a} \in A^i$ such that $f^i\left(G^{-1}(z), \bar{a}\right) = J^{-1}_\varepsilon(z)p$, and we obtain that $\tilde{f}^i_\varepsilon(z, \bar{a}) = p$.

Let us now define the counterparts of $\mathcal{M}_\varepsilon$, f_ε and ℓ_ε:

$$\mathcal{M} = \left\{(x,a);\ x \in \mathbb{R}^2, \quad a \in A^i \text{ if } x \in \Omega^i, \text{ and } a \in A^L \cup A^R \text{ if } x \in \Gamma\right\}, \tag{2.6}$$

$$\forall (z,a) \in \mathcal{M}, \qquad \tilde{f}_\varepsilon(z,a) = \begin{cases} \tilde{f}^i_\varepsilon(z,a) & \text{if } x \in \Omega^i, \\ \tilde{f}^i_\varepsilon(z,a) & \text{if } x \in \Gamma \text{ and } a \in A^i, \end{cases} \tag{2.7}$$

$$\forall (z,a) \in \mathcal{M}, \qquad \tilde{\ell}_\varepsilon(z,a) = \begin{cases} \tilde{\ell}^i_\varepsilon(z,a) & \text{if } x \in \Omega^i, \\ \tilde{\ell}^i_\varepsilon(z,a) & \text{if } x \in \Gamma \text{ and } a \in A^i. \end{cases} \tag{2.8}$$

For $x \in \mathbb{R}^2$, the set of admissible trajectories starting from x is

$$\widetilde{\mathcal{T}}_{x,\varepsilon} = \left\{ \begin{array}{ll} (y_x, a) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}) : & y_x \in \mathrm{Lip}(\mathbb{R}^+; \mathbb{R}^2), \\ & y_x(t) = x + \displaystyle\int_0^t \tilde{f}_\varepsilon(y_x(s), a(s))ds \quad \forall t \in \mathbb{R}^+ \end{array} \right\}. \tag{2.9}$$

Note that $\forall z \in \mathbb{R}^2$, $(y_z, a) \in \widetilde{\mathcal{T}}_{z,\varepsilon} \Leftrightarrow \left(G^{-1}(y_z(\cdot)), a\right) \in \mathcal{T}_{G^{-1}(z),\varepsilon}$. The new optimal control problem consists in finding

$$\tilde{v}_\varepsilon(z) = \inf_{(y_z,a)\in\widetilde{\mathcal{T}}_{z,\varepsilon}} \int_0^\infty \tilde{\ell}_\varepsilon(y_z(t), a(t))e^{-\lambda t}dt. \tag{2.10}$$

Remark 2.1 (Relationship between v_ε and $\tilde{v}_\varepsilon$)**.** *For any $z \in \mathbb{R}^2$,*

$$\tilde{v}_\varepsilon(z) = v_\varepsilon(G^{-1}(z)) = v_\varepsilon\left(z_1 + \varepsilon g\left(\frac{z_2}{\varepsilon}\right), z_2\right).$$

2.3 The Hamilton-Jacobi equation in the straightened geometry

2.3.1 Hamiltonians

If $i \in \{L, R\}$, the Hamiltonians $\tilde{H}^i_\varepsilon : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$\tilde{H}^i_\varepsilon(z,p) = \max_{a\in A^i}\left(-\tilde{f}^i_\varepsilon(z,a)\cdot p - \tilde{\ell}^i_\varepsilon(z,a)\right) = \max_{a\in A^i}\left(-J_\varepsilon(z)f^i(G^{-1}(z),a)\cdot p - \ell^i(G^{-1}(z),a)\right). \tag{2.11}$$

More explicitly,

$$\tilde{H}^i_\varepsilon(z,p) = \max_{a\in A^i}\left(-\begin{pmatrix}1 & -g'(\frac{z_2}{\varepsilon}) \\ 0 & 1\end{pmatrix} f^i((z_1 + \varepsilon g(\frac{z_2}{\varepsilon}), z_2), a)\cdot p - \ell^i((z_1 + \varepsilon g(\frac{z_2}{\varepsilon}), z_2), a)\right).$$

If $z \in \Gamma$, the Hamiltonian $\tilde{H}_{\Gamma,\varepsilon} : \Gamma \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\tilde{H}_{\Gamma,\varepsilon}(z, p^L, p^R) = \max\left(\tilde{H}^{+,L}_{\Gamma,\varepsilon}(z, p^L), \tilde{H}^{+,R}_{\Gamma,\varepsilon}(z, p^R)\right), \tag{2.12}$$

where for $i = 1, 2$, $z \in \mathbb{R}^2$, $p^i \in \mathbb{R}^2$, and σ^i is defined in § 1.1,

$$\tilde{H}^{+,i}_{\Gamma,\varepsilon}(z,p) = \max_{\substack{a \in A^i \text{ s.t.} \\ \sigma^i \tilde{f}^i_\varepsilon(z,a)\cdot e_1 \ge 0}} (-\tilde{f}^i_\varepsilon(z,a)\cdot p - \tilde{\ell}^i_\varepsilon(z,a)).$$

If $z \in \Gamma$, then $n_\varepsilon(G^{-1}(z)) = J_\varepsilon(z)^T e_1$, by (2.2). Hence, $\tilde{H}^{+,i}_{\Gamma,\varepsilon}$ is the counterpart of $H^{+,i}_{\Gamma,\varepsilon}$.

2.3.2 Definition of viscosity solutions in the straightened geometry

Definition 2.2. *The function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an admissible test-function for the fixed geometry if ϕ is continuous in $\mathbb{R}^2$ and for any $i = L, R$, $\phi|_{\bar{\Omega}^i} \in \mathcal{C}^1(\bar{\Omega}^i)$.*
The set of the admissible test-functions is denoted $\mathcal{R}$. If $\phi \in \mathcal{R}$, $x \in \Gamma$ and $i \in \{L, R\}$, we set $D\phi^i(x) = \lim_{\substack{x' \to x \\ x' \in \Omega^i}} D\phi(x')$. Of course, the partial derivatives of $\phi|_{\bar{\Omega}^L}$ and $\phi|_{\bar{\Omega}^R}$ with respect to x_2 coincide on Γ.

We then define the sub/super-solutions and solutions of

$$\lambda u + \tilde{\mathcal{H}}_\varepsilon(z, Du) = 0 \tag{2.13}$$

as in Definition 1.3, using the set of test-functions $\mathcal{R}$, the Hamiltonians $\tilde{H}^i_\varepsilon(z, p)$ if $z \in \Omega^i$ and $\tilde{H}_{\Gamma,\varepsilon}(z, p^L, p^R)$ if $z \in \Gamma$.

Remark 2.3. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an upper semi-continuous (resp. lower semi-continuous) function and $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\tilde{u}(z) = u(G^{-1}(z))$. Then u is a subsolution (resp. supersolution) of* (1.14) *if and only if $\tilde{u}$ is a subsolution (resp. supersolution) of* (2.13)*.*

2.3.3 Existence and uniqueness

We have seen in Remark 2.1 that the optimal control problems (1.10) and (2.10) are equivalent; similarly Remark 2.3 tells us that the notions of viscosity solutions of (1.14) and (2.13) are equivalent. Therefore, it is enough to focus on (2.10) and (2.13).

Lemma 2.4. *There exists $r > 0$ such that any bounded viscosity subsolution u of* (1.14) *(resp.* (2.13)*) is Lipschitz continuous in $B(\Gamma_\varepsilon, r)$ (resp. $B(\Gamma, r)$) with Lipschitz constant $L_u \le \frac{\lambda\|u\|_\infty + M_\ell}{\delta_0}$ (resp. $L_u \le \sqrt{2}\frac{(\lambda\|u\|_\infty + M_\ell)(1+\|g'\|_\infty)}{\delta_0}$), where for X a closed subset of $\mathbb{R}^2$, $B(X, r)$ denotes the set $\{y \in \mathbb{R}^2 : dist(y, X) < r\}$.*

Proof. For a subsolution u of (2.13), the result is exactly [16, Lemma 2.6].
If u is a subsolution of (1.14), then $\tilde{u}(z) = u(G^{-1}(z))$ is a subsolution of (2.13), and is therefore Lipschitz continuous in a neighborhood of Γ. Since $u = \tilde{u} \circ G$, u is Lipschitz continuous in a neighborhood of Γ_ε. □

Theorem 2.5 (Local comparison principle)**.** *Let u be a bounded viscosity subsolution of* (1.14) *(resp.* (2.13)*), and v be a bounded viscosity supersolution of* (1.14) *(resp.* (2.13)*). For any $z \in \mathbb{R}^2$, there exists $r > 0$ such that*

$$\| (u - v)_+ \|_{L^\infty(B(z,r))} \le \| (u - v)_+ \|_{L^\infty(\partial B(z,r))} . \tag{2.14}$$

Proof. Let us focus on (2.13). If $z \in \Omega^i$, then we can choose $r > 0$ small enough so that $B(z, r) \subset \Omega^i$ and the result is classical. If $z \in \Gamma$, the result stems from a direct application of [16, Theorem 3.3]. Indeed, all the assumptions required by [16, Theorem 3.3] are satisfied thanks to the properties $[\widetilde{\mathrm{H}}0]_\varepsilon$-$[\widetilde{\mathrm{H}}3]_\varepsilon$. The result for (1.14) can be deduced from the latter thanks to Remark 2.3. □

Theorem 2.6 (Global comparison principle)**.** *Let u be a bounded viscosity subsolution of* (1.14) *(resp.* (2.13)*), and v be a bounded viscosity supersolution of* (1.14) *(resp.* (2.13)*). Then $u \le v$.*

Proof. The result for equation (2.13) stems from a direct application of [16, Theorem 3.4]. Then we deduce the result for equation (1.14) thanks to Remark 2.3. □

Theorem 2.7. *The value function v_ε (resp. $\tilde{v}_\varepsilon$) defined in* (1.10) *(resp.* (2.10)*) is the unique bounded viscosity solution of* (1.14) *(resp.* (2.13)*).*

Proof. Uniqueness is a direct consequence of Theorem 2.6 for both equations (1.14) and (2.13). Existence for equation (2.13) is proved in the same way as in [16, Theorem 2.3]. Then, existence for (1.14) is deduced from Remark 2.3. □

Remark 2.8. *Under suitable assumptions, see §4 in [16], all the above results hold if we modify* (1.14) *(resp.*(2.13)*) by adding to H_{Γ_ε} (resp. $\tilde{H}_{\Gamma,\varepsilon}$) a Hamiltonian H^0_ε (resp. $\tilde{H}^0_\varepsilon$) correponding to trajectories staying on the junctions.*

3 Asymptotic behavior in Ω^L and Ω^R

Our goal is to understand the asymptotic behavior of the sequence $(v_\varepsilon)_\varepsilon$ as ε tends to 0. In this section, we are going to see that the Hamilton-Jacobi equations remain unchanged in Ω^L and Ω^R; this is not surprising because the amplitude of the oscillations of the interface vanishes as $\varepsilon \to 0$. Then, we are going to introduce some of the ingredients of the effective boundary conditions on Γ.
From Remark 2.1, the sequence $(v_\varepsilon)_\varepsilon$ converges if and only if the sequence $(\tilde{v}_\varepsilon)_\varepsilon$ converges. Moreover, if they converge, the two sequences have the same limit. It will be convenient to focus on the asymptotic behavior of the sequence $(\tilde{v}_\varepsilon)_\varepsilon$, since the geometry is fixed. It is now classical to consider the relaxed semi-limits

$$\overline{\tilde{v}}(z) = \limsup_{\varepsilon}{}^* \tilde{v}_\varepsilon(z) = \limsup_{z'\to z,\varepsilon\to 0} \tilde{v}_\varepsilon(z') \quad \text{and} \quad \underline{\tilde{v}}(z) = \liminf_{\varepsilon}{}_* \tilde{v}_\varepsilon(z) = \liminf_{z'\to z,\varepsilon\to 0} \tilde{v}_\varepsilon(z'). \tag{3.1}$$

Note that $\overline{\tilde{v}}$ and $\underline{\tilde{v}}$ are well defined, since $(\tilde{v}_\varepsilon)_\varepsilon$ is uniformly bounded by $\frac{M_\ell}{\lambda}$, see (2.10).

3.1 For the Hamilton-Jacobi equations in Ω^L and Ω^R, nothing changes

Proposition 3.1. *For $i = L, R$, the functions $\overline{\tilde{v}}(z)$ and $\underline{\tilde{v}}(z)$ are respectively a bounded subsolution and a bounded supersolution in Ω^i of*

$$lu(z) + H^i(z, Du(z)) = 0, \tag{3.2}$$

where the Hamiltonian H^i is given by (1.11).

Proof. The proof is classical and relies on perturbed test-functions techniques, see[8]. For a test-function ϕ (near a point $\bar{z}$ for example), the main idea is to construct the perturbed test-function $\phi_\varepsilon(z) = \phi(z) + \varepsilon\partial_{z_1}\phi(\bar{z})g(\frac{z_2}{\varepsilon}) - \delta$, for a suitable positive number δ. □

3.2 An ingredient in the effective transmission condition on Γ: the Hamiltonian H_Γ inherited from the half-planes

For $i \in \{L, R\}$, let us define the Hamiltonian $H^{+,i}$ and $H^{-,i}$: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H^{\pm,i}(z,p) = \max_{a\in A^i \text{ s.t. } \pm\sigma^i f^i(z,a).e_1 \geq 0} (-p.f^i(z,a) - \ell^i(z,a)). \tag{3.3}$$

and $H_\Gamma : \Gamma \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H_\Gamma(z, p^L, p^R) = \max\left(H^{+,L}(z,p^L), H^{+,R}(z,p^R)\right). \tag{3.4}$$

As in [3, 10], we introduce the functions $E_0^i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = L, R$ and $E_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$
\begin{aligned}
E_0^i(z_2, p_2) &= \min \left\{ H^i((0, z_2), p_2 e_2 + q e_1), \quad q \in \mathbb{R} \right\}, && (3.5)\\
E_0(z_2, p_2) &= \max \left\{ E_0^L(z_2, p_2), E_0^R(z_2, p_2) \right\}. && (3.6)
\end{aligned}
$$

The following lemma, which is the same as [16, Lemma 2.1], deals with some monotonicity properties of $H^{\pm,i}$:

Lemma 3.2.

1. *For any $(0, z_2) \in \Gamma$, $p \in \mathbb{R}^2$, $p_1 \mapsto H^{+,L}((0, z_2), p + p_1 e_1)$ and $p_1 \mapsto H^{-,R}((0, z_2), p + p_1 e_1)$ are nondecreasing; $p_1 \mapsto H^{-,L}((0, z_2), p + p_1 e_1)$ and $p_1 \mapsto H^{+,R}((0, z_2), p + p_1 e_1)$ are nonincreasing.*

2. *For $z_2, p_2 \in \mathbb{R}$, there exist two unique real numbers $p_0^{-,L}(z_2, p_2) \le p_0^{+,L}(z_2, p_2)$ such that*

$$
H^{-,L}((0, z_2), p_2 e_2 + p e_1) = \begin{cases} H^L((0, z_2), p_2 e_2 + p e_1) & \text{if } p \le p_0^{-,L}(z, p_2), \\ E_0^L(z_2, p_2) & \text{if } p > p_0^{-,L}(z, p_2), \end{cases}
$$

$$
H^{+,L}((0, z_2), p_2 e_2 + p e_1) = \begin{cases} E_0^L(z_2, p_2) & \text{if } p \le p_0^{+L}(z, p_2), \\ H^L((0, z_2), p_2 e_2 + p e_1) & \text{if } p > p_0^{+,L}(z, p_2). \end{cases}
$$

3. *For $z_2, p_2 \in \mathbb{R}$, there exist two unique real numbers $p_0^{+,R}(z_2, p_2) \le p_0^{-,R}(z_2, p_2)$ such that*

$$
H^{-,R}((0, z_2), p_2 e_2 + p e_1) = \begin{cases} E_0^R(z_2, p_2) & \text{if } p \le p_0^{-,R}(z, p_2), \\ H^R((0, z_2), p_2 e_2 + p e_1) & \text{if } p > p_0^{-,R}(z, p_2), \end{cases}
$$

$$
H^{+,R}((0, z_2), p_2 e_2 + p e_1) = \begin{cases} H^R((0, z_2), p_2 e_2 + p e_1) & \text{if } p \le p_0^{+,R}(z, p_2), \\ E_0^R(z_2, p_2) & \text{if } p > p_0^{+,R}(z, p_2). \end{cases}
$$

4 A new Hamiltonian involved in the effective transmission condition

In this section, we construct the effective Hamiltonian corresponding to effective dynamics staying on the interface Γ, by using similar ideas as those presented in [3]. We will define an effective Hamiltonian E on Γ as the limit of a sequence of ergodic constants for state-constrained problems in larger and larger truncated domains. We will also construct correctors associated to the effective Hamiltonian. The noteworthy difficulty is that the correctors need to be defined in an unbounded domain.

4.1 Fast and slow variables

Let us introduce the fast variable $y_2 = \frac{z_2}{\varepsilon}$. Neglecting the contribution of $\varepsilon g(y_2)$ in the Hamiltonians $\tilde{H}_\varepsilon^i$ previously defined in (2.11), we obtain the new Hamiltonians $\tilde{H}^i : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$:

$$
\tilde{H}^i(z, p, y_2) := \max_{a \in A^i} \left(-\tilde{J}(y_2) f^i(z, a) \cdot p - \ell^i(z, a) \right), \tag{4.1}
$$

where

$$
\tilde{J}(y_2) = \begin{pmatrix} 1 & -g'(y_2) \\ 0 & 1 \end{pmatrix}. \tag{4.2}
$$

As above, using σ^i introduced in § 1.1, we also define $\tilde{H}^{+,i}$ and $\tilde{H}^{-,i}$: $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{H}^{\pm,i}(z,p,y_2) = \max_{\substack{a \in A^i \text{ s.t.} \\ \pm\sigma^i \tilde{J}(y_2) f^i(z,a)\cdot e_1 \geq 0}} \Big(-\tilde{J}(y_2) f^i(z,a)\cdot p - \ell^i(z,a)\Big). \tag{4.3}$$

Lemma 4.1. *With L_f, M_f, M_ℓ and δ_0 appearing in Assumptions* [H0]-[H3]*, for any $z \in \mathbb{R}^2$, $y_2 \in \mathbb{R}$ and $p, p' \in \mathbb{R}^2$,*

$$|\tilde{H}^i(z,p,y_2) - \tilde{H}^i(z,p',y_2)| \leq M_f |p-p'|, \quad i = L, R, \tag{4.4}$$

and there exists a constant $M > 0$ (which can be computed from L_f, M_f, δ_0 and $\|g'\|_\infty$) and a modulus of continuity ω (which can be deduced from ω_ℓ, M_ℓ L_f, δ_0 and $\|g'\|_\infty$) such that for any $z, z' \in \mathbb{R}^2$, $y_2 \in \mathbb{R}$ and $p \in \mathbb{R}^2$,

$$|\ \tilde{H}^i(z,p,y_2) - \tilde{H}^i(z',p,y_2)\ | \leq M|p||z-z'| + \omega(|z-z'|). \tag{4.5}$$

Similar estimates hold for $\tilde{H}^{+,i}$ and $\tilde{H}^{-,i}$.

Proof. The proof is standard for the Hamiltonians $\tilde{H}^i$. Adapting the proofs of Lemmas 3.5 and 3.6 in [16], we see that similar estimates hold for $\tilde{H}^{+,i}$ and $\tilde{H}^{-,i}$: the proofs are not direct and rely on the convexity of $\{(\tilde{J}(y_2)f^i(z,a), \ell^i(z,a)), a \in A^i\}$, on the fact that the dynamics $\tilde{J}(y_2)f^i(z,\cdot)$ satisfy a strong controlability property similar to [H3] uniformly w.r.t. z, and on continuity properties of $(z,a) \mapsto \tilde{J}(y_2)f^i(z,a)$ and $(z,a) \mapsto \ell^i(z,a)$ similar to [H0] and [H1]. □

Remark 4.2. *Take $z \in \Omega^i$ and $p \in \mathbb{R}^2$. The unique real number $\lambda^i(z,p)$ such that the following one dimensional cell problem in the variable y_2*

$$\begin{cases} \tilde{H}^i(z, p + \chi'(y_2)e_2, y_2) = \lambda^i(z,p), \\ \chi \textit{ is 1-periodic w.r.t. } y_2, \end{cases} \tag{4.6}$$

admits a viscosity solution is $\lambda^i(z,p) = H^i(z,p)$; indeed, for this choice of $\lambda^i(z,p)$, it is easy to check that $\chi(y_2) = p_1 g(y_2)$ is a solution of (4.6) *and the uniqueness of $\lambda^i(z,p)$ such that* (4.6) *has a solution is well known, see e.g. [15, 8].*

Remark 4.3. *For any $(0,z_2) \in \Gamma$, $p \in \mathbb{R}^2$ and $y_2 \in \mathbb{R}$, the functions $p_1 \mapsto \tilde{H}^{\pm,i}((0,z_2), p + p_1e_1, y_2)$ have the same monotonicity properties as those stated in point 1 in Lemma 3.2 for $p_1 \mapsto H^{\pm,i}((0,z_2), p + p_1e_1)$. Similarly, one can prove the counterparts of points 2 and 3 in Lemma 3.2 involving $\tilde{H}^{\pm,i}((0,z_2), p + p_1e_1, y_2)$, $\tilde{H}^i((0,z_2), p + p_1e_1, y_2)$ and*

$$\tilde{E}_0^i(z_2, p_2, y_2) = \min\left\{\tilde{H}^i((0,z_2), p_2e_2 + qe_1, y_2),\ q \in \mathbb{R}\right\}.$$

4.2 Ergodic constants for state-constrained problems in truncated domains

4.2.1 State-constrained problem in truncated domains

Let us fix $z = (0,z_2) \in \Gamma$ and $p_2 \in \mathbb{R}$. For $\rho > 0$, we consider the *truncated cell problem*:

$$\begin{cases} \tilde{H}^L((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\rho(z_2,p_2) & \text{in } (-\rho,0)\times\mathbb{R}, \\ \tilde{H}^R((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\rho(z_2,p_2) & \text{in } (0,\rho)\times\mathbb{R}, \\ \max\limits_{i=L,R}\Big(\tilde{H}^{+,i}((0,z_2), Du^i(y) + p_2e_2, y_2)\Big) & = \lambda_\rho(z_2,p_2) & \text{on } \Gamma, \\ \tilde{H}^{-,L}((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\rho(z_2,p_2) & \text{on } \{-\rho\}\times\mathbb{R}, \\ \tilde{H}^{-,R}((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\rho(z_2,p_2) & \text{on } \{\rho\}\times\mathbb{R}, \\ u \text{ is 1-periodic w.r.t. } y_2, \end{cases} \tag{4.7}$$

where the Hamiltonians $\tilde{H}^i$, $\tilde{H}^{+,i}$ and $\tilde{H}^{-,i}$ are respectively defined in (4.1) and (4.3). The notions of viscosity subsolution, supersolution and solution of (4.7) are defined in the same way as in Definition 1.3 using the set of test-functions

$$\mathcal{R}_\rho = \left\{\psi|_{[-\rho,\rho]\times\mathbb{R}},\ \psi \in \mathcal{R}\right\}, \tag{4.8}$$

with $\mathcal{R}$ defined in Definition 2.2. The following stability property allows one to construct a solution of (4.7):

Lemma 4.4 (A stability result)**.** *Let $(u^\eta)_\eta$ be a sequence of uniformly Lipschitz continuous solutions of the perturbed equation*

$$\begin{cases} \eta u(y) + \tilde{H}^L((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\eta \quad \text{in } (-\rho,0)\times\mathbb{R}, \\ \eta u(y) + \tilde{H}^R((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\eta \quad \text{in } (0,\rho)\times\mathbb{R}, \\ \eta u(y) + \max\limits_{i=L,R}\left(\tilde{H}^{+,i}((0,z_2), Du^i(y) + p_2e_2, y_2)\right) & = \lambda_\eta \quad \text{on } \Gamma, \\ \eta u(y) + \tilde{H}^{-,L}((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\eta \quad \text{on } \{-\rho\}\times\mathbb{R}, \\ \eta u(y) + \tilde{H}^{-,R}((0,z_2), Du(y) + p_2e_2, y_2) & = \lambda_\eta \quad \text{on } \{\rho\}\times\mathbb{R}, \\ u \text{ is 1-periodic w.r.t. } y_2, \end{cases} \tag{4.9}$$

such that λ_η tends to λ as η tends to 0 and u^η converges to u^0 uniformly in $[-\rho,\rho]\times\mathbb{R}$. Then u^0 is a viscosity solution of (4.7) (replacing $\lambda_\rho(z_2,p_2)$ with λ).

Proof. The proof of Lemma 4.4 follows the lines of the proofs of Theorem 6.1 and Theorem 6.2 in [2]. Actually, the proof is even simpler in the present case since the involved Hamiltonians do not depend of η. We give it in Appendix A for the reader's convenience. □

The following comparison principle for (4.9) yields the uniqueness of the constant $\lambda_\rho(z_2,z_2)$ for which the cell-problem (4.7) admits a solution:

Lemma 4.5 (A comparison result)**.** *For $\eta > 0$, let u be a bounded subsolution of (4.9) and v be a bounded supersolution of (4.9). Then $u \le v$ in $[-\rho,\rho]\times\mathbb{R}$.*

Proof. As for Theorem 2.6, this result can be obtained by applying [16, Theorem 3.4]. □

Lemma 4.6. *There is a unique $\lambda_\rho(z_2,p_2)\in\mathbb{R}$ such that (4.7) admits a bounded solution. For this choice of $\lambda_\rho(z_2,p_2)$, there exists a solution $\chi_\rho(z_2,p_2,\cdot)$ which is Lipschitz continuous with Lipschitz constant L depending on p_2 only (independent of ρ).*

Proof. With the set $\mathcal{M}$ defined in (2.6), let us consider the new *freezed* dynamics $f_{z_2}:\mathcal{M}\to\mathbb{R}^2$ and running costs $\ell_{z_2,p_2}:\mathcal{M}\to\mathbb{R}^2$:

$$f_{z_2}(y,a) \;=\; \begin{cases} \begin{pmatrix} 1 & -g'(y_2) \\ 0 & 1 \end{pmatrix} f^L((0,z_2),a) & \text{if } y_1 \le 0, a \in A^L, \\ \begin{pmatrix} 1 & -g'(y_2) \\ 0 & 1 \end{pmatrix} f^R((0,z_2),a) & \text{if } y_1 \ge 0, a \in A^R, \end{cases} \tag{4.10}$$

$$\ell_{z_2,p_2}(y,a) \;=\; \begin{cases} f_2^L((0,z_2),a)p_2 + \ell^L((0,z_2),a) & \text{if } y_1 \le 0, a \in A^L, \\ f_2^R((0,z_2),a)p_2 + \ell^R((0,z_2),a) & \text{if } y_1 \ge 0, a \in A^R, \end{cases} \tag{4.11}$$

where f_2 stands for the second component of f.
Let $\mathcal{T}_{z_2,x,\rho}$ be the set of admissible trajectories starting from $y \in (-\rho,\rho)\times\mathbb{R}$ and constrained to $[-\rho,\rho]\times\mathbb{R}$:

$$\mathcal{T}_{z_2,y,\rho} = \left\{ \begin{array}{ll} (\zeta_y,a)\in L^\infty_{\text{loc}}(\mathbb{R}^+;\mathcal{M}): & \zeta_y \in \text{Lip}(\mathbb{R}^+;[-\rho,\rho]\times\mathbb{R}), \\ & \zeta_y(t) = y + \displaystyle\int_0^t f_{z_2}(\zeta_y(s),a(s))ds \quad \forall t\in\mathbb{R}^+ \end{array} \right\}. \tag{4.12}$$

For any $\eta > 0$, the cost associated to the trajectory $(\zeta_y, a) \in \mathcal{T}_{z_2,y,\rho}$ is

$$\mathcal{J}_\rho^\eta(z_2, p_2, y; (\zeta_y, a)) = \int_0^\infty \ell_{z_2,p_2}(\zeta_y(t), a(t))e^{-\eta t}dt, \tag{4.13}$$

and we introduce the optimal control problem:

$$v_\rho^\eta(z_2, p_2, y) = \inf_{(\zeta_y,a)\in\mathcal{T}_{z_2,y,\rho}} \mathcal{J}_\rho^\eta(z_2, p_2, y; (\zeta_y, a)). \tag{4.14}$$

Thanks to [H3], we see that if $\delta_0' = \frac{\delta_0}{\sqrt{2}(1+\|g'\|_\infty)}$, then $B(0, \delta_0') \subset \{f_{z_2}(y, a), a \in A^i\}$ for any $i = L, R$, $y \in [-\rho, \rho] \times \mathbb{R}$. This strong controlability property can be proved in the same manner as $[\widetilde{\mathrm{H}}3]_\varepsilon$ in § 2.2. From this, it follows that for any $y, y' \in [-\rho, \rho] \times \mathbb{R}$,

$$|v_\rho^\eta(z_2, p_2, y) - v_\rho^\eta(z_2, p_2, y')| \leq L(p_2)|y - y'|. \tag{4.15}$$

for some $L(p_2) = L_1 + L_2|p_2|$ with L_1, L_2 depending on M_f, M_ℓ δ_0 and $\| g' \|_\infty$ but not on p_2. Introducing $\chi_\rho^\eta(z_2, p_2, y) = v_\rho^\eta(z_2, p_2, y) - v_\rho^\eta(z_2, p_2, (0, 0))$, we deduce from (4.14) and (4.15) that there exists a constant $K = K(p_2)$ such that

$$|\eta v_\rho^\eta(z_2, p_2, y)| \leq K, \quad \text{and} \quad |\chi_\rho^\eta(z_2, p_2, y)| \leq K. \tag{4.16}$$

From Ascoli-Arzela's theorem, up to the extraction a subsequence, $\chi_\rho^\eta(z_2, p_2, \cdot)$ and $-\eta v_\rho^\eta(z_2, p_2, \cdot)$ converge uniformly respectively to a Lipschitz function $\chi_\rho(z_2, p_2, \cdot)$ defined on $[-\rho, \rho] \times \mathbb{R}$ (with Lipschitz constant L) and to a constant $\lambda_\rho(z_2, p_2)$ as $\eta \to 0$.
On the other hand, with the arguments contained in [1, 11, 12], it can be proved that $v_\rho^\eta(z_2, p_2, \cdot)$ is a viscosity solution of (4.9) with $\lambda_\eta = 0$. Hence, $\chi_\rho^\eta(z_2, p_2, \cdot)$ is a sequence of viscosity solutions of (4.9) for $\lambda_\eta = -\eta v_\rho^\eta(z_2, p_2, (0, 0))$, and $\lambda_\eta \to \lambda_\rho(z_2, p_2)$ as η tends to 0. From the stability result in Lemma 4.4, the function $\chi_\rho(z_2, p_2, \cdot)$ is a viscosity solution of (4.7). Finally, uniqueness can be proved in a classical way using the comparison principle in Lemma 4.5 and the boundedness of χ_ρ. □

4.2.2 Passage to the limit as $\rho \to +\infty$

By definition of $\mathcal{T}_{z_2,y,\rho}$, it is clear that if $\rho_1 \leq \rho_2$, then $\mathcal{T}_{z_2,y,\rho_1} \subset \mathcal{T}_{z_2,y,\rho_2}$. Then, thanks to (4.14) and (4.16), we see that

$$-\eta v_{\rho_1}^\eta \leq -\eta v_{\rho_2}^\eta \leq K,$$

and letting $\eta \to 0$, we obtain that

$$\lambda_{\rho_1}(z_2, p_2) \leq \lambda_{\rho_2}(z_2, p_2) \leq K. \tag{4.17}$$

Definition 4.7. *We define the effective tangential Hamiltonian $E(z_2, p_2)$ as*

$$E(z_2, p_2) = \lim_{\rho\to\infty} \lambda_\rho(z_2, p_2). \tag{4.18}$$

For $z_2, p_2 \in \mathbb{R}$ fixed, we consider the *global cell-problem*

$$\begin{cases} \tilde{H}^i((0, z_2), Du(y) + p_2e_2, y_2) & = E(z_2, p_2) \quad \text{in } \Omega^i, \\ \max\Big(\tilde{H}^{+,L}((0, z_2), Du^L(y) + p_2e_2, y_2), \tilde{H}^{+,R}((0, z_2), Du^R(y) + p_2e_2, y_2)\Big) & = E(z_2, p_2) \quad \text{on } \Gamma, \\ u \text{ is 1-periodic w.r.t. } y_2, \end{cases} \tag{4.19}$$

The following stability result is useful for proving the existence of a viscosity solution u of the cell-problem (4.19):

Lemma 4.8. *Let u_ρ be a sequence of uniformly Lipschitz continuous solutions of the truncated cell-problem* (4.7) *which converges to u locally uniformly on $\mathbb{R}^2$. Then u is a viscosity solution of the global cell-problem* (4.19)*.*

Proof. Proceed exactly in the same way as in the proof of Lemma 4.4. □

Theorem 4.9 (Existence of a global corrector)**.** *There exists $\chi(z_2, p_2, \cdot)$ a Lipschitz continuous viscosity solution of*(4.19) *with the same Lipschitz constant L as in* (4.15) *and such that* $\chi(z_2, p_2, (0, 0)) = 0$.

Proof. Let $\chi_\rho(z_2, p_2, \cdot)$ be the sequence of solutions of (4.7) given by Lemma 4.6. Recall that $\chi_\rho(z_2, p_2, \cdot)$ is Lipschitz continuous with Lipschitz constant L independent of ρ and periodic with respect to y_2. By taking $\chi_\rho(z_2, p_2, \cdot) - \chi_\rho(z_2, p_2, (0, 0))$ instead of $\chi_\rho(z_2, p_2, \cdot)$, we may assume that $\chi_\rho(z_2, p_2, (0, 0)) = 0$. Thus, $\chi_\rho(z_2, p_2, \cdot)$ is locally bounded and thanks to Ascoli-Arzela's theorem, up to the extraction a subsequence, $\chi_\rho(z_2, p_2, \cdot)$ converges locally uniformy to a function $\chi(z_2, p_2, \cdot)$, which is Lipschitz continuous and periodic with respect to y_2 and satisfies $\chi(z_2, p_2, (0, 0)) = 0$. Thanks to the stability result in Lemma 4.8, $\chi(z_2, p_2, \cdot)$ is a viscosity solution of (4.19). □

4.2.3 Comparison between E_0 and E respectively defined in (3.6) and (4.18)

For $\varepsilon > 0$, let us call $W_\varepsilon(z_2, p_2, y) = \varepsilon\chi(z_2, p_2, \frac{y}{\varepsilon})$. The following result is reminiscent of [10, Theorem 4.6,iii]:

Lemma 4.10. *For any $z_2, p_2 \in \mathbb{R}$, there exists a subsequence ε_n such that $W_{\varepsilon_n}(z_2, p_2, \cdot)$ converges locally uniformly to a Lipschitz function $y \mapsto W(z_2, p_2, y)$, with the Lipschitz constant L appearing in* (4.15)*. This function is constant with respect to y_2 and satisfies $W(z_2, p_2, 0) = 0$. It is a viscosity solution of*

$$H^i((0, z_2), Du(y) + p_2 e_2) = E(z_2, p_2) \qquad \text{in } \Omega^i. \tag{4.20}$$

Proof. It is clear that $y \mapsto W_\varepsilon(z_2, p_2, y)$ is a Lipschitz continuous function with constant L and that $W_\varepsilon(z_2, p_2, (0, 0)) = 0$. Thus, from Ascoli-Arzela's Theorem, we may assume that $y \mapsto W_\varepsilon(z_2, p_2, y)$ converges locally uniformly to some function $y \mapsto W(z_2, p_2, y)$, up to the extraction of subsequences. The function $y \mapsto W(z_2, p_2, y)$ is Lipschitz continuous with constant L and $W(z_2, p_2, (0, 0)) = 0$. Moreover, since $W_\varepsilon(z_2, p_2, y)$ is periodic with respect to y_2 with period ε, $W(z_2, p_2, y)$ does not depend on y_2. To prove that $W(z_2, p_2, \cdot)$ is a viscosity solution of (4.20), we first observe that $y \mapsto W_\varepsilon(z_2, p_2, \cdot)$ is a viscosity solution of

$$\tilde{H}^i((0, z_2), Du(y) + p_2 e_2, \frac{y_2}{\varepsilon}) = E(z_2, p_2) \quad \text{in } \Omega^i. \tag{4.21}$$

For $i = L, R$, assume that $\bar{y} \in \Omega^i$, $\phi \in \mathcal{C}^1(\Omega^i)$ and $r_0 < 0$ are such that $B(\bar{y}, r_0) \subset \Omega^i$ and that

$$W(z_2, p_2, y) - \phi(y) < W(z_2, p_2, \bar{y}) - \phi(\bar{y}) = 0 \text{ for } y \in B(\bar{y}, r_0) \setminus \{\bar{y}\}.$$

We wish to prove that $H^i((0, z_2), D\phi(\bar{y}) + p_2 e_2) \le E(z_2, p_2)$. Let us argue by contradiction and assume that there exists $\theta > 0$ such that

$$H^i((0, z_2), D\phi(\bar{y}) + p_2 e_2) = E(z_2, p_2) + \theta. \tag{4.22}$$

Take $\phi_\varepsilon(y) = \phi(y) + \varepsilon\partial_{y_1}\phi(\bar{y})g(\frac{y_2}{\varepsilon}) - \delta$, where $\delta > 0$ is a fixed positive number. We claim that for $\varepsilon > 0$ and $r > 0$ small enough, ϕ_ε is a viscosity supersolution of

$$\tilde{H}^i((0, z_2), Du(y) + p_2 e_2, \frac{y_2}{\varepsilon}) \ge E(z_2, p_2) + \frac{\theta}{2} \qquad \text{in } B(\bar{y}, r). \tag{4.23}$$

Indeed ϕ_ε is a regular function which satisfies

$$\tilde{H}^i((0,z_2), D\phi_\varepsilon(y) + p_2 e_2, \frac{y_2}{\varepsilon}) = H^i\left((0,z_2), D\phi(y) + g'(\frac{y_2}{\varepsilon})(\partial_{y_1}\phi(\bar{y}) - \partial_{y_1}\phi(y))e_2\right),$$

and we deduce (4.23) from (4.22) and the regularity properties of the Hamiltonian H^i. Hence, $W_\varepsilon(z_2,p_2,\cdot)$ is a subsolution of (4.21) and ϕ_ε is a supersolution of (4.23) in $B(\bar{y},r)$. Moreover for $r>0$ small enough, $\max_{y\in\partial B(\bar{y},r)}(W(z_2,p_2,y)-\phi(y))<0$. Hence, for $\varepsilon>0$ small enough $\max_{y\in\partial B(\bar{y},r)}(W_\varepsilon(z_2,p_2,y)-\phi_\varepsilon(y))\le 0$.
Thanks to a standard comparison principle (which holds thanks to the fact that $\frac{\theta}{2}>0$)

$$\max_{y\in B(\bar{y},r)}(W_\varepsilon(z_2,p_2,y)-\phi_\varepsilon(y))\le 0. \tag{4.24}$$

Letting $\varepsilon\to 0$ in (4.24), we deduce that $W(z_2,p_2,\bar{y})\le\phi(\bar{y})-\delta$, which is in contradiction with the assumption. $\square$

Using Lemma 4.10, it is possible to compare $E_0(z_2,p_2)$ and $E(z_2,p_2)$ respectively defined in (3.6) and (4.18):

Proposition 4.11. *For any $z_2,p_2\in\mathbb{R}$,*

$$E(z_2,p_2)\ge E_0(z_2,p_2). \tag{4.25}$$

Proof. Let $i\in\{L,R\}$ be fixed. Thanks to Lemma 4.10, the function $y\mapsto W(z_2,p_2,y)$ is a viscosity solution of (4.20) in Ω^i. Therefore, (4.20) is satisfied by $W(z_2,p_2,\cdot)$ almost everywhere. Keeping in mind that $W(z_2,p_2,y)$ is independent of y_2, we see that for almost all $y\in\Omega^i$, $E(z_2,p_2)=H^i((0,z_2),\partial_{y_1}W(z_2,p_2,y_1)e_1+p_2e_2)\ge E_0^i(z_2,p_2)$. $\square$

From Proposition 4.11 and the coercivity of the Hamiltonian H^i, the following numbers are well defined for all $z_2,p_2\in\mathbb{R}$:

$$\overline{\Pi}^L(z_2,p_2)=\max\left\{p\in\mathbb{R}: H^L((0,z_2),p_2e_2+pe_1)=H^{-,L}((0,z_2),p_2e_2+pe_1)=E(z_2,p_2)\right\} \tag{4.26}$$
$$\widehat{\Pi}^L(z_2,p_2)=\min\left\{p\in\mathbb{R}: H^L((0,z_2),p_2e_2+pe_1)=H^{-,L}((0,z_2),p_2e_2+pe_1)=E(z_2,p_2)\right\} \tag{4.27}$$
$$\overline{\Pi}^R(z_2,p_2)=\min\left\{p\in\mathbb{R}: H^R((0,z_2),p_2e_2+pe_1)=H^{-,R}((0,z_2),p_2e_2+pe_1)=E(z_2,p_2)\right\} \tag{4.28}$$
$$\widehat{\Pi}^R(z_2,p_2)=\max\left\{p\in\mathbb{R}: H^R((0,z_2),p_2e_2+pe_1)=H^{-,R}((0,z_2),p_2e_2+pe_1)=E(z_2,p_2)\right\} \tag{4.29}$$

Remark 4.12. *In § 5, see in particular Corollaries 5.5 and 5.6 below, we will see that the function W which is defined in Lemma 4.10 and provides information on the growth of $y\mapsto\chi(z_2,p_2,y)$ as $|y_1|\to\infty$, satisfies*

$$\overline{\Pi}^L(z_2,p_2)y_1 1_{\Omega^L}+\overline{\Pi}^R(z_2,p_2)y_1 1_{\Omega^R}\le W(z_2,p_2,y)\le\widehat{\Pi}^L(z_2,p_2)y_1 1_{\Omega^L}+\widehat{\Pi}^R(z_2,p_2)y_1 1_{\Omega^R}.$$

These growth properties at infinity show that $\chi(z_2,p_2,\cdot)$ is precisely the corrector associated to the reduced set of test-functions proposed by Imbert and Monneau in [11, 12], see § 7.1 below.

Remark 4.13. *From the convexity of the Hamiltonians H^i and $H^{-,i}$, we deduce that if $E_0^i(z_2,p_2)<E(z_2,p_2)$, then $\overline{\Pi}^i(z_2,p_2)=\widehat{\Pi}^i(z_2,p_2)$. In this case, we will use the notation*

$$\Pi^i(z_2,p_2)=\overline{\Pi}^i(z_2,p_2)=\widehat{\Pi}^i(z_2,p_2). \tag{4.30}$$

5 Further properties of the correctors

In this section, we prove further growth properties of the correctors, which will be useful in the proof of convergence in § 7 below, see Remark 4.12. We start by stating a useful comparison principle related to a mixed boundary value problem:

Lemma 5.1. *Take $0 < \rho_1 < \rho_2$, $z_2, p_2, \lambda \in \mathbb{R}$, a continuous function $U_0 : \mathbb{R} \to \mathbb{R}$ and $\varepsilon_0 > 0$. Let v be a continuous viscosity supersolution of*

$$\begin{cases} \tilde{H}^R((0, z_2), Dv(y) + p_2e_2, y_2) \geq \lambda, & y = (y_1, y_2) \in (\rho_1, \rho_2) \times \mathbb{R}, \\ \tilde{H}^{-,R}((0, z_2), Dv(y) + p_2e_2, y_2) \geq \lambda, & y = (y_1, y_2) \in \{\rho_2\} \times \mathbb{R}, \\ v(y) \geq U_0(y_2), & y = (y_1, y_2) \in \{\rho_1\} \times \mathbb{R}, \\ v \text{ is 1-periodic w.r.t. } y_2, \end{cases} \tag{5.1}$$

and u be a continuous viscosity subsolution of

$$\begin{cases} \tilde{H}^R((0, z_2), Du(y) + p_2e_2, y_2) \leq \lambda - \varepsilon_0, & y = (y_1, y_2) \in (\rho_1, \rho_2) \times \mathbb{R}, \\ \tilde{H}^{-,R}((0, z_2), Du(y) + p_2e_2, y_2) \leq \lambda - \varepsilon_0, & y = (y_1, y_2) \in \{\rho_2\} \times \mathbb{R}, \\ u(y) \leq U_0(y_2), & y = (y_1, y_2) \in \{\rho_1\} \times \mathbb{R}, \\ u \text{ is 1-periodic w.r.t. } y_2, \end{cases} \tag{5.2}$$

where the inequalities on $y_1 = \rho_1$ are understood pointwise. Then, $u \leq v$ in $[\rho_1, \rho_2] \times \mathbb{R}$.

The proof is rather classical, and follows the lines of [4] Theorem IV.5.8. We skip it for brevity.

Remark 5.2. *From [11, Proposition 2.14], we know that a bounded lsc function v is a supersolution of* (5.1) *if and only if it is a supersolution of*

$$\begin{cases} \tilde{H}^R((0, z_2), Dv(y) + p_2e_2, y_2) \geq \lambda, & y = (y_1, y_2) \in (\rho_1, \rho_2] \times \mathbb{R}, \\ v(y) \geq U_0(y_2), & y = (y_1, y_2) \in \{\rho_1\} \times \mathbb{R}, \\ v \text{ is 1-periodic w.r.t. } y_2, \end{cases}$$

and that a bounded usc function u is a subsolution of (5.2) *if and only if it is a subsolution of*

$$\begin{cases} \tilde{H}^R((0, z_2), Du(y) + p_2e_2, y_2) \leq \lambda - \varepsilon_0, & y = (y_1, y_2) \in (\rho_1, \rho_2) \times \mathbb{R}, \\ u(y) \leq U_0(y_2), & y = (y_1, y_2) \in \{\rho_1\} \times \mathbb{R}, \\ u \text{ is 1-periodic w.r.t. } y_2. \end{cases}$$

In other words, the boundary conditions on $y_1 = \rho_2$ correspond to state constraints.

Remark 5.3. *There is of course a similar result for the mirror boundary value problem posed in $[-\rho_2, -\rho_1] \times \mathbb{R} \subset \Omega^L$ with the Hamiltonian $\tilde{H}^L$ instead of $\tilde{H}^R$, a Dirichlet condition on $y_1 = -\rho_1$ and a state constrained boundary condition on $y_1 = -\rho_2$ (i.e. involving $\tilde{H}^{-,L}$).*

Proposition 5.4 (Control of slopes on the truncated domain)**.** *With E and E_0^R respectively defined in (4.18) and (3.5), let $z_2, p_2 \in \mathbb{R}$ be such that $E(z_2, p_2) > E_0^R(z_2, p_2)$. There exists $\rho^* = \rho^*(z_2, p_2) > 0$, $\delta^* = \delta^*(z_2, p_2) > 0$, $m(\cdot) = m(z_2, p_2, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\delta\to 0^+} m(\delta) = 0$ and $M^* = M^*(z_2, p_2)$ such that for all $\delta \in (0, \delta^*]$, $\rho \geq \rho^*$, $y = (y_1, y_2) \in [\rho^*, \rho] \times \mathbb{R}$, $h_1 \in [0, \rho - y_1]$ and $h_2 \in \mathbb{R}$,*

$$\chi_\rho(z_2, p_2, y + h_1e_1 + h_2e_2) - \chi_\rho(z_2, p_2, y) \geq (\Pi^R(z_2, p_2) - m(\delta))h_1 - M^*, \tag{5.3}$$

where $\Pi^R(z_2,p_2)$ is given by (4.30) and $\chi_\rho(z_2,p_2,\cdot)$ is a solution of (4.7) given by Lemma 4.6.

Similarly, let $z_2,p_2 \in \mathbb{R}$ be such that $E(z_2,p_2) > E_0^L(z_2,p_2)$. There exists $\rho^* > 0$, $\delta^* > 0$, $m(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and M^* as above, such that for all $\delta \in (0,\delta^*]$, $\rho \geq \rho^*$, $y = (y_1,y_2) \in [-\rho,-\rho^*] \times \mathbb{R}$, $h_1 \in [0,\rho+y_1]$ and $h_2 \in \mathbb{R}$,

$$\chi_\rho(z_2,p_2,y-h_1e_1+h_2e_2) - \chi_\rho(z_2,p_2,y) \geq -(\Pi^L(z_2,p_2)+m(\delta))h_1 - M^*. \tag{5.4}$$

Proof. Let us focus on (5.3) since the proof of (5.4) is similar. Recall that $\rho \mapsto \lambda_\rho(z_2,p_2)$ is nondecreasing and tends to $E(z_2,p_2)$ as $\rho \to +\infty$. Choose $\rho^* = \rho^*(z_2,p_2) > 0$ such that for any $\rho \geq \rho^*$, $E(z_2,p_2) > \lambda_\rho(z_2,p_2) > E_0^R(z_2,p_2)$. Then, choose $\delta^* = \delta^*(z_2,p_2) > 0$ such that for any $\delta \in (0,\delta^*]$, $\lambda_\rho(z_2,p_2) - \delta > E_0^R(z_2,p_2)$.

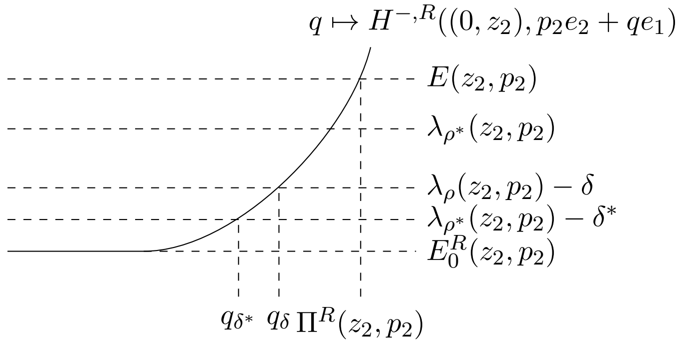

Figure 2: Construction of ρ^*, δ^* and q_δ

Let us fix $\rho > \rho^*$, $\delta \in (0,\delta^*]$ and $\bar{y} = (\bar{y}_1,\bar{y}_2) \in [\rho^*,\rho] \times \mathbb{R}$. Consider $y \mapsto \chi_\rho(z_2,p_2,y)$ a solution of (4.7) as in Lemma 4.6. The function $\chi_\rho(z_2,p_2,\cdot)$ is 1-periodic with respect to y_2 and Lipschitz continuous with constant $L = L(p_2)$. Thus, for any $y \in \{\bar{y}_1\} \times \mathbb{R}$

$$\chi_\rho(z_2,p_2,y) - \chi_\rho(z_2,p_2,\bar{y}) \geq -L.$$

Therefore, $y \mapsto \chi_\rho(z_2,p_2,y) - \chi_\rho(z_2,p_2,\bar{y})$ is a supersolution of

$$\begin{cases} \tilde{H}^R((0,z_2),Dv(y)+p_2e_2,y_2) \geq \lambda_\rho(z_2,p_2), & y \in (\bar{y}_1,\rho) \times \mathbb{R}, \\ \tilde{H}^{-,R}((0,z_2),Dv(y)+p_2e_2,y_2) \geq \lambda_\rho(z_2,p_2), & y \in \{\rho\} \times \mathbb{R}, \\ v(y) \geq -L, & y \in \{\bar{y}_1\} \times \mathbb{R}, \\ v \text{ is 1-periodic w.r.t. } y_2. \end{cases} \tag{5.5}$$

Since $\rho \geq \rho^*$ and $\delta \in (0,\delta^*]$, there exists a unique $q_\delta \in \mathbb{R}$, see Figure 2, such that

$$\lambda_\rho(z_2,p_2) - \delta = H^R((0,z_2),p_2e_2+q_\delta e_1) = H^{-,R}((0,z_2),p_2e_2+q_\delta e_1). \tag{5.6}$$

We observe that $q_{\delta^*} \leq q_\delta \leq \Pi^R(z_2,p_2)$ and that q_δ tends to $\Pi^R(z_2,p_2)$ as δ tends to 0. Choose $m(\delta) = \Pi^R(z_2,p_2) - q_\delta \geq 0$ and consider the function w^R: $w^R(y) = q_\delta(y_1+g(y_2))$, which is of class $\mathcal{C}^2$. From the choice of q_δ, for any $y \in \mathbb{R}^2$,

$$\tilde{H}^{-,R}(z_2,Dw^R(y)+p_2e_2,y_2) \leq \tilde{H}^R(z_2,Dw^R(y)+p_2e_2,y_2) = \lambda_\rho(z_2,p_2) - \delta, \tag{5.7}$$

and for any $y \in \{\bar{y}_1\} \times \mathbb{R}$,

$$w^R(y) - q_\delta\bar{y}_1 \leq |q_\delta| \parallel g \parallel_\infty \leq \max\left(|q_{\delta^*}|, |\Pi^R(z_2,p_2)|\right) \parallel g \parallel_\infty = C = C(z_2,p_2).$$

Therefore, as stated in Remark 5.2, the subsolution property holds up to the boundary $y_1 = \rho$ and the function $u^R : y \in [\bar{y}_1, R] \times \mathbb{R} \mapsto w^R(y) - q_\delta \bar{y}_1 - C - L$ is a subsolution of

$$\begin{cases} \tilde{H}^R((0,z_2), Du(y) + p_2 e_2, y_2) \le \lambda_\rho(z_2,p_2) - \delta & y \in (\bar{y}_1, \rho) \times \mathbb{R}, \\ \tilde{H}^{-,R}((0,z_2), Du(y) + p_2 e_2, y_2) \le \lambda_\rho(z_2,p_2) - \delta, & y \in \{\rho\} \times \mathbb{R}, \\ u(y) \le -L, & y \in \{\bar{y}_1\} \times \mathbb{R}, \\ u \text{ is 1-periodic w.r.t. } y_2. \end{cases} \tag{5.8}$$

Finally, since $\chi_\rho(z_2, p_2, y) - \chi_\rho(z_2, p_2, \bar{y})$ is a supersolution of (5.5) and u^R is a subsolution of (5.8), the comparison principle in Lemma 5.1 yields: for all $y \in [\bar{y}_1, \rho] \times \mathbb{R}$

$$\begin{aligned} \chi_\rho(z_2,p_2,y) - \chi_\rho(z_2,p_2,\bar{y}) \ge u^R(y) &= q_\delta((y_1 - \bar{y}_1) + g(y_2)) - C - L \\ &\ge (\Pi^R(z_2,p_2) - m(\delta))(y_1 - \bar{y}_1) - M^*, \end{aligned} \tag{5.9}$$

where M^* is a constant depending only of z_2 and p_2. Note that the constants which appear in (5.9) are independent of $\rho > 0$. $\square$

The following corollary deals with the global corrector χ:

Corollary 5.5. *With $\Pi^i(z_2,p_2)$ defined in (4.30), $i = L, R$,*

1. *If $E(z_2,p_2) > E_0^R(z_2,p_2)$, then, with $\rho^* > 0$ and $M^* \in \mathbb{R}$ as in the first point of Proposition 5.4, for all $y \in [\rho^*, +\infty) \times \mathbb{R}$, $h_1 \ge 0$ and $h_2 \in \mathbb{R}$,*

$$\chi(z_2,p_2,y + h_1e_1 + h_2e_2) - \chi(z_2,p_2,y) \ge \Pi^R(z_2,p_2)h_1 - M^*. \tag{5.10}$$

2. *If $E(z_2,p_2) > E_0^L(z_2,p_2)$, then, with $\rho^* > 0$ and $M^* \in \mathbb{R}$ as in the second point of Proposition 5.4, for all $y \in (-\infty, -\rho^*] \times \mathbb{R}$, $h_1 \ge 0$ and $h_2 \in \mathbb{R}$,*

$$\chi(z_2,p_2,y - h_1e_1 + h_2e_2) - \chi(z_2,p_2,y) \ge -\Pi^L(z_2,p_2)h_1 - M^*. \tag{5.11}$$

Proof. The proof follows easily from Proposition 5.4 and the local uniform convergence of the sequence $\chi_\rho(z_2,p_2,\cdot)$ toward $\chi(z_2,p_2,\cdot)$. $\square$

Corollary 5.6. *For $z_2, p_2 \in \mathbb{R}$, $y \mapsto W(z_2,p_2,y)$ defined in Lemma 4.10 satisfies*

$$\overline{\Pi}^R(z_2,p_2) \le \partial_{y_1} W(z_2,p_2,y) \le \widehat{\Pi}^R(z_2,p_2) \qquad \text{for a.a. } y \in (0,+\infty) \times \mathbb{R}, \tag{5.12}$$

$$\widehat{\Pi}^L(z_2,p_2) \le \partial_{y_1} W(z_2,p_2,y) \le \overline{\Pi}^L(z_2,p_2) \qquad \text{for a.a. } y \in (-\infty,0) \times \mathbb{R}, \tag{5.13}$$

and for all y:

$$\overline{\Pi}^L(z_2,p_2)y_1 1_{\Omega^L} + \overline{\Pi}^R(z_2,p_2)y_1 1_{\Omega^R} \le W(z_2,p_2,y) \le \widehat{\Pi}^L(z_2,p_2)y_1 1_{\Omega^L} + \widehat{\Pi}^R(z_2,p_2)y_1 1_{\Omega^R}. \tag{5.14}$$

Proof. From Lemma 4.10, we see that $y \mapsto W(z_2,p_2,y)$ is Lipschitz continuous w.r.t. y_1 and independent of y_2, and satisfies

$$H^R((0,z_2), \partial_{y_1} W(z_2,p_2,y)e_1 + p_2e_2) = E(z_2,p_2) \quad \text{a.e. in } \Omega^R. \tag{5.15}$$

Consider first the case when $E(z_2,p_2) > E_0^R(z_2,p_2)$; from the convexity and coercivity of H^R, the observations above yield that almost everywhere in y, $\partial_{y_1} W(z_2,p_2,y)$ can be either $\Pi^R(z_2,p_2)$ (the unique real number such that $H^{-,R}((0,z_2), qe_1 + p_2e_2) = E(z_2,p_2)$), or the unique real number q (depending on (z_2,p_2)) such that $H^{+,R}((0,z_2), qe_1 + p_2e_2) = E(z_2,p_2)$. Note that

$q < \Pi^R(z_2, p_2)$. But from Corollary 5.5 and the local uniform convergence of $W_\varepsilon(z_2, p_2, \cdot)$ toward $W(z_2, p_2, \cdot)$, we see that that for any $y_1 > 0$ and $h_1 \geq 0$,

$$W(z_2, p_2, y + h_1 e_1) - W(z_2, p_2, y) \geq \Pi^R(z_2, p_2) h_1,$$

which implies that almost everywhere, $\partial_{y_1} W(z_2, p_2, y) \geq \Pi^R(z_2, p_2) > q$. Therefore, $\partial_{y_1} W(z_2, p_2, \cdot) = \Pi^R(z_2, p_2)$ almost everywhere.
In the case when $E(z_2, p_2) = E_0^R(z_2, p_2)$, we deduce from (5.15) that almost everywhere in y, $\overline{\Pi}^R(z_2, p_2) \leq \partial_{y_1} W(z_2, p_2, y) \leq \widehat{\Pi}^R(z_2, p_2)$.
We have proved (5.12). The proof of (5.13) is identical. Finally, (5.14) comes from (5.12), (5.13) and from the fact that $W(z_2, p_2, 0) = 0$. $\square$

6 A comparison principle for (1.21)

To prove the main result of the paper, i.e. Theorem 1.5, we need a comparison principle for (1.19)-(1.20). Before proving such a result, we need to establish some useful properties of E arising in (1.20).

6.1 Properties of $E(\cdot, \cdot)$

In the theory of homogenization of Hamilton-Jacobi equations, it is quite standard to observe that the effective Hamiltonian inherits some properties from the original problem, see the pioneering work [15].

Lemma 6.1. *For any $z_2 \in \mathbb{R}$, the function $p_2 \mapsto E(z_2, p_2)$ is convex. For any $z_2, z_2', p_2, p_2' \in \mathbb{R}$,*

$$| E(z_2, p_2) - E(z_2, p_2') | \leq M_f |p_2 - p_2'|, \tag{6.1}$$

$$| E(z_2, p_2) - E(z_2', p_2) | \leq C(1 + |p_2|)|z_2 - z_2'| + \omega(|z_2 - z_2'|), \tag{6.2}$$

$$\delta_0 |p_2| - M_\ell \leq E(z_2, p_2) \leq M_f |p_2| + M_\ell, \tag{6.3}$$

where the constants M_f, M_ℓ, δ_0 have been introduced in Assumptions [H0]-[H3]*, the modulus of continuity ω has been introduced in Lemma 4.1 and C is a positive constant.*
Moreover, $p_2 \mapsto E(z_2, p_2)$ is affine in a neighborhood of $\pm\infty$. More precisely, for any $z_2 \in \mathbb{R}$, there exist $\hat{\ell}(z_2), \check{\ell}(z_2) \in [-M_\ell, M_\ell]$, $\hat{f}(z_2), \check{f}(z_2) \in [\delta_0, M_f]$ and $\hat{K}(z_2), \check{K}(z_2) > 0$ such

$$p_2 \geq \hat{K}(z_2) \quad \Rightarrow \quad E(z_2, p_2) = \hat{f}(z_2) p_2 + \hat{\ell}(z_2), \tag{6.4}$$

$$p_2 \leq -\check{K}(z_2) \quad \Rightarrow \quad E(z_2, p_2) = -\check{f}(z_2) p_2 + \check{\ell}(z_2). \tag{6.5}$$

Proof. The proof contains arguments that are quite similar to those contained in [15], but technical difficulties arise from the discontinuities of the Hamiltonians at $y_1 = 0$. The main idea is to deduce the desired properties from those of $-\eta v_\rho^\eta$, where v_ρ^η is defined in (4.14). For brevity, we only prove (6.2) and that $p_2 \mapsto E(z_2, p_2)$ is affine in a neighborhood of $\pm\infty$.

Proof of (6.2) For $z_2, z_2', p_2 \in \mathbb{R}$, consider $y \mapsto v_\rho^\eta(z_2, p_2, y)$ and $y \mapsto v_\rho^\eta(z_2', p_2, y)$ given by (4.14). These functions are viscosity solutions of (4.9) with $\lambda_\eta = 0$. Assume that $0 = v_\rho^\eta(z_2', p_2, \bar{y}) - \varphi(\bar{y})$ be a local minimum of $v_\rho^\eta(z_2', p_2, \cdot) - \varphi(\cdot)$ for $\bar{y} \in \mathbb{R}^2$ and $\varphi \in \mathcal{R}_\rho$. As above, we focus on the case when $\bar{y} \in \Gamma$ because the other cases are simpler. It is not restrictive to assume that φ^L and φ^R are smooth (at least $\mathcal{C}^3$). Since $y \mapsto v_\rho^\eta(z_2', p_2, y)$ is Lipschitz continuous with a constant $L(p_2) = L_1 + L_2|p_2|$, see (4.15), we see that

$$|\partial_{y_2} \varphi^L(\bar{y})| = |\partial_{y_2} \varphi^R(\bar{y})| \leq L(p_2), \quad \partial_{y_1} \varphi^L(\bar{y}) \geq -L(p_2), \quad \partial_{y_1} \varphi^R(\bar{y}) \leq L(p_2). \tag{6.6}$$

It is always possible to modify φ and obtain a test-function ψ such that $|\partial_{y_1}\psi^i(\bar y)| \le 2L(p_2)$, $|\partial_{y_2}\psi^i(\bar y)| \le L(p_2)$, $i = L, R$ and that $v^\eta_\rho(z_2', p_2, \cdot) - \psi(\cdot)$ has a local minimum at $\bar y$. Indeed, we make out two cases:

1. if $|\partial_{y_1}\varphi^i(\bar y)| \le 2L(p_2)$ for $i = L, R$, then we choose $\psi = \varphi$.

2. If $\partial_{y_1}\varphi^L(\bar y) > 2L(p_2)$ or $\partial_{y_1}\varphi^R(\bar y) < -2L(p_2)$, let us introduce

$$\psi(y) = \begin{cases} \varphi(y) - (2L(p_2) + \partial_{y_1}\varphi^R(\bar y))y_1 - A|y-\bar y|^2 & \text{if } y \in [0,\rho]\times\mathbb{R} \text{ and } \partial_{y_1}\varphi^R(\bar y) < -2L(p_2),\\ \varphi(y) - A|y-\bar y|^2 & \text{if } y \in [0,\rho]\times\mathbb{R} \text{ and } |\partial_{y_1}\varphi^R(\bar y)| \le 2L(p_2),\\ \varphi(y) + (2L(p_2) - \partial_{y_1}\varphi^L(\bar y))y_1 - A|y-\bar y|^2 & \text{if } y \in [-\rho,0]\times\mathbb{R} \text{ and } \partial_{y_1}\varphi^L(\bar y) > 2L(p_2),\\ \varphi(y) - A|y-\bar y|^2 & \text{if } y \in [-\rho,0]\times\mathbb{R} \text{ and } |\partial_{y_1}\varphi^L(\bar y)| \le 2L(p_2). \end{cases}$$

Note that $|\partial_{y_1}\psi^i(\bar y)| \le 2L(p_2)$, $i = L, R$ and that

$$\partial_{y_1}\varphi^L(\bar y) \ge \partial_{y_1}\psi^L(\bar y), \quad \text{and} \quad \partial_{y_1}\varphi^R(\bar y) \le \partial_{y_1}\psi^R(\bar y). \tag{6.7}$$

We claim that for A large enough, $v^\eta_\rho(z_2', p_2, \cdot) - \psi(\cdot)$ has a local minimum at $\bar y$. Indeed, fix $r > 0$ such that $v^\eta_\rho(z_2', p_2, y) - \varphi(y) \ge v^\eta_\rho(z_2', p_2, \bar y) - \varphi(\bar y) = 0$ for $y \in B(\bar y, r)$. Assuming for example that $\partial_{y_1}\varphi^R(\bar y) < -2L(p_2)$ (the case when $|\partial_{y_1}\varphi^R(\bar y)| \le 2L(p_2)$ is obvious), we see that for $y \in B(\bar y, r)$ with $y_1 > 0$,

$$\begin{aligned} v^\eta_\rho(z_2', p_2, y) - \psi(y) \ge & v^\eta_\rho(z_2', p_2, y) - v^\eta_\rho(z_2', p_2, (0, y_2)) + \varphi(0, y_2) - \varphi(y) \\ & + \left(2L(p_2) + \partial_{y_1}\varphi^R(\bar y)\right) y_1 + A|y-\bar y|^2. \end{aligned} \tag{6.8}$$

For a constant $c > 0$ depending on ϕ and r,

$$\varphi(0, y_2) - \varphi(y) \quad \ge \quad -y_1\partial_{y_1}\varphi^R(0, y_2) - c y_1^2 \ge -y_1\partial_{y_1}\varphi^R(\bar y) - c(y_1^2 + y_1|y_2 - \bar y_2|).$$

On the other hand, $v^\eta_\rho(z_2', p_2, y) - v^\eta_\rho(z_2', p_2, (0, y_2)) \ge -L(p_2)y_1$. From the latter two observations and (6.8), we deduce that $v^\eta_\rho(z_2', p_2, y) - \psi(y) \ge L(p_2)y_1 - c(y_1^2 + y_1|y_2 - \bar y_2|) + A|y-\bar y|^2$. Therefore for A large enough,

$$v^\eta_\rho(z_2', p_2, y) - \psi(y) \ge 0 = v^\eta_\rho(z_2', p_2, \bar y) - \psi(\bar y), \quad y \in B(\bar y, r),$$

and the claim is proved.

In both cases, we see that $\eta v^\eta_\rho(z_2', p_2, \bar y) + \max_{i\in\{L,R\}}\left(\tilde H^{+,i}((0, z_2'), D\psi^i(\bar y) + p_2 e_2, \bar y_2)\right) \ge 0$; assuming that the latter maximum is achieved by $i = R$ for example, this yields

$$\begin{aligned} & \eta\left(v^\eta_\rho(z_2', p_2, \bar y) + \tfrac{C}{\eta}(1+|p_2|)|z_2 - z_2'| + \tfrac{1}{\eta}\omega(|z_2 - z_2'|)\right) + \tilde H^{+,R}((0, z_2), D\psi^R(\bar y) + p_2 e_2, \bar y_2) \\ & \ge C(1+|p_2|)|z_2 - z_2'| - M|p_2 e_2 + D\psi^R(\bar y)||z_2 - z_2'|. \end{aligned}$$

Thanks to (6.6) and from the construction of ψ,

$$|p_2 e_2 + D\psi^i(\bar y)| \le |p_2| + 3L(p_2), \quad i = L, R. \tag{6.9}$$

Hence, choosing $C \ge M\max(3L_1, 1 + 3L_2)$ yields

$$\eta\left(v^\eta_\rho(z_2', p_2, \bar y) + \tfrac{C}{\eta}(1+|p_2|)|z_2 - z_2'| + \tfrac{1}{\eta}\omega(|z_2 - z_2'|)\right) + \tilde H^{+,R}((0, z_2), D\psi^R(\bar y) + p_2 e_2, \bar y_2) \ge 0.$$

But from (6.7) and the nonincreasing character of $\tilde H^{+,R}$, we see that

$$\eta\left(v^\eta_\rho(z_2', p_2, \bar y) + \tfrac{C}{\eta}(1+|p_2|)|z_2 - z_2'| + \tfrac{1}{\eta}\omega(|z_2 - z_2'|)\right) + \tilde H^{+,R}((0, z_2), D\varphi^R(\bar y) + p_2 e_2, \bar y_2) \ge 0.$$

Therefore, $v^\eta_\rho(z_2', p_2, \cdot) + \frac{C}{\eta}(1+|p_2|)|z_2 - z_2'| + \frac{1}{\eta}\omega(|z_2 - z_2'|)$ is a supersolution of the equation satisfied by $v^\eta_\rho(z_2, p_2, \cdot)$ and we conclude using the comparison principle Lemma 4.5, passing to the limit as $\eta \to 0$ and $\rho \to +\infty$, and finally exchanging the roles of z_2 and z_2'.

Proof that $p_2 \mapsto E(z_2, p_2)$ is affine in a neighborhood of $\pm\infty$ We focus on (6.4) since the proof of (6.5) is similar. For $z_2 \in \mathbb{R}$, $y \in [-\rho, \rho] \times \mathbb{R}$ and $p_2, \eta, \rho > 0$, let us define $\bar{f}^\eta_\rho(z_2, y) = \sup_{(\gamma_y, a) \in \mathcal{T}_{z_2,y,\rho}} \left\{ -\int_0^\infty f_2((0, z_2), a(t)) e^{-\eta t} dt \right\}$, with $\mathcal{T}_{z_2,y,\rho}$ given in (4.12). From (4.11) and (4.14), we deduce that

$$\eta \bar{f}^\eta_\rho(z_2, y) p_2 - M_\ell \le -\eta v^\eta_\rho(z_2, p_2, y) \le \eta \bar{f}^\eta_\rho(z_2, y) p_2 + M_\ell. \tag{6.10}$$

From the assumptions, it is easy to check that

$$|\eta \bar{f}^\eta_\rho(z_2, y)| \le M_f, \tag{6.11}$$

$$|\bar{f}^\eta_\rho(z_2, y) - \bar{f}^\eta_\rho(z_2, y')| \le C|y - y'|, \tag{6.12}$$

for some positive constant C, and that $y \mapsto \bar{f}^\eta_\rho(z_2, y)$ is periodic with period 1 in the variable y_2. From Ascoli-Arzela theorem, up to the extraction of subsequences, we may assume that $\bar{f}^\eta_\rho(z_2, \cdot) - \bar{f}^\eta_\rho(z_2, 0)$ tends to a Lipschitz function (with Lipschitz constant C) and that $\eta \bar{f}^\eta_\rho(z_2, \cdot)$ tends to a constant $\bar{f}_\rho(z_2)$ as $\eta \to 0$. With the same arguments as in § 4.2.2, we may prove that $\bar{f}_\rho(z_2)$ is nondecreasing and uniformly bounded with respect to ρ. Therefore, we may define $\hat{f}(z_2) = \lim_{\rho \to +\infty} \bar{f}_\rho(z_2)$.
Passing to the limit in (6.10) as $\eta \to 0$ then as $\rho \to +\infty$, we deduce that

$$\hat{f}(z_2) p_2 - M_\ell \le E(z_2, p_2) \le \hat{f}(z_2) p_2 + M_\ell. \tag{6.13}$$

Finally, from (6.13) and the convexity of $p_2 \mapsto E(z_2, p_2)$, we infer that there exists $\hat{\ell}(z_2) \in [-M_\ell, M_\ell]$ and $\hat{K}(z_2) > 0$ such that for any $p_2 \ge \hat{K}(z_2)$

$$E(z_2, p_2) = \hat{f}(z_2) p_2 + \hat{\ell}(z_2).$$

Finally, the bound $\hat{f}(z_2) \ge \delta_0$ comes from (6.3), and it is simple to check that $\hat{f}(z_2) \le M_f$. □

6.2 The comparison principles

Since we are not able to control the constants $\hat{K}(z_2)$ and $\check{K}(z_2)$ arising in Lemma 6.1, we cannot directly use the comparison principle which is available in [16, Theorem 2.5]. To apply the latter, it will be useful to first modify $E(z_2, p_2)$ for $|p_2|$ larger than some fixed number K independent of z_2. The following lemma deals with such modified Hamiltonians.

Lemma 6.2. *For a positive number K, the Hamiltonian $E_K(z_2, p_2)$ defined by*

$$E_K(z_2, p_2) = \begin{cases} E(z_2, p_2) & \text{if} \quad |p_2| \le K, \\ E(z_2, K) + M_f(p_2 - K) & \text{if} \quad p_2 > K, \\ E(z_2, -K) - M_f(p_2 + K) & \text{if} \quad p_2 < -K, \end{cases} \tag{6.14}$$

is convex in the variable p_2 and

$$E_K(z_2, p_2) = \max_{b \in [-M_f, M_f]} \left(b p_2 - E^\star_K(z_2, b) \right), \tag{6.15}$$

where $E^\star_K : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ is the Fenchel transform $E^\star_K(z_2, b) = \sup_{q \in \mathbb{R}} (bq - E_K(z_2, q))$. For $z_2, z'_2 \in \mathbb{R}$ and $b, b' \in [-M_f, M_f]$,

$$|E^\star_K(z_2, b)| \le C_K, \tag{6.16}$$

$$|E^\star_K(z_2, b) - E^\star_K(z_2, b')| \le K|b - b'|, \tag{6.17}$$

$$|E^\star_K(z_2, b) - E^\star_K(z'_2, b)| \le \omega_K(t) = C(1 + K)|z_2 - z'_2| + \omega(|z_2 - z'_2|), \tag{6.18}$$

where in (6.16)*, C_K is a positive constant, and, in* (6.18)*, the constant C and the modulus of continuity ω are those appearing in (6.2).*

Proof. The convexity of $p_2 \mapsto E_K(z_2,p_2)$ comes from the convexity of $p_2 \mapsto E(z_2,p_2)$ and from (6.1). From (6.1), it is also clear that $E_K^\star(z_2,b) = +\infty$ if $b \notin [-M_f, M_f]$, which implies (6.15). It can also be seen that if $b \in [-M_f, M_f]$, then

$$E_K^\star(z_2,b) = \max_{p\in[-K,K]} \left(bp - E_K(z_2,p)\right). \tag{6.19}$$

From (6.3), we check that for $z_2 \in \mathbb{R}$, $p_2 \in [-K,K]$ and $b \in [-M_f, M_f]$,

$$bp_2 - M_f|p_2| - M_\ell \le bp_2 - E_K(z_2,p_2) \le bp_2 + M_f(K - |p_2|) - \delta_0 K + M_\ell. \tag{6.20}$$

Choosing $p_2 = 0$ yields that $E_K^\star(z_2,b_2) \ge -M_\ell$. Using the fact that $bp_2 - M_f|p_2| \le 0$ if $|b| \le M_f$, we deduce that $E_K^\star(z_2,b) \le C_K$ and finally (6.16) with $C_K = (M_f - \delta_0)K + M_\ell$.
It is standard to deduce (6.17) and (6.18) from (6.19) and (6.2). □

Lemma 6.2 allows us to prove the following comparison principle:

Proposition 6.3. *Let u and w be respectively a bounded subsolution and a bounded supersolution of*

$$\begin{array}{ll} \lambda u(z) + H^i(z, Du(z)) = 0 & \text{if } z \in \Omega^i, \\ \lambda u(z) + \max\left(E_K(z_2, \partial_{z_2}\varphi(z)), H_\Gamma(z, Du^L(z), Du^R(z))\right) = 0 & \text{if } z = (0,z_2) \in \Gamma. \end{array} \tag{6.21}$$

Then, $u \le w$ in $\mathbb{R}^2$.

Proof. Thanks to Lemma 6.2, it is possible to apply [16, Theorem 2.5], more precisely the general comparison principle discussed in [16, Remark 2.11], since the continuity of u is not assumed. □

Theorem 6.4. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ be respectively a bounded subsolution and a bounded supersolution of* (1.21)*. Then $u \le v$ in $\mathbb{R}^2$.*

Proof. Since u is a subsolution of (1.21), it is also a subsolution of

$$\begin{array}{rl} \lambda u(z) + H^i(z, Du(z)) = 0 & \text{if } z \in \Omega^i, \\ \lambda u(z) + H_\Gamma(z, Du^L(z), Du^R(z)) = 0 & \text{if } z = (0,z_2) \in \Gamma. \end{array}$$

Thanks to Assumptions [H0]-[H3], we can apply [16, Lemma 2.6] to u: there exists $r > 0$ such that u is Lipschitz continuous in $[-r,r] \times \mathbb{R}$. Let us call L_u the Lipschitz constant of the restriction of u to $[-r,r] \times \mathbb{R}$ and choose $K \ge L_u$. Since E_K coincides with E on $\mathbb{R} \times [-K,K]$, we deduce that u is a subsolution of (6.21).
On the other hand, since $E_K \ge E$, v is a supersolution of (6.21).
The proof is achieved by applying Proposition 6.3 to the pair (u,v). □

7 Proof of the main result

7.1 A reduced set of test-functions

From [11] and [12], we may use an equivalent definition for the viscosity solution of (1.21).

Definition 7.1. *Recall that* $\overline{\Pi}^i$ *and* $\widehat{\Pi}^i$, $i = L, R$, *have been introduced in* (4.26)-(4.29). *Let* $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$, $(z_2, p_2) \mapsto \big(\Pi^L(z_2,p_2), \Pi^R(z_2,p_2)\big)$ *be such that, for all* (z_2, p_2)

$$\begin{aligned} \widehat{\Pi}^L(z_2,p_2) \le \Pi^L(z_2,p_2) \le \overline{\Pi}^L(z_2,p_2), \\ \overline{\Pi}^R(z_2,p_2) \le \Pi^R(z_2,p_2) \le \widehat{\Pi}^R(z_2,p_2). \end{aligned} \tag{7.1}$$

For $\bar z = (0, \bar z_2) \in \Gamma$, *the reduced set of test-functions* $\mathcal{R}^\Pi(\bar z)$ *associated to the map* Π *is the set of the functions* $\varphi \in \mathcal{C}^0(\mathbb{R}^2)$ *such that there exists a* $\mathcal{C}^1$ *function* $\psi : \mathbb{R} \to \mathbb{R}$ *with*

$$\varphi(z) = \psi(z_2) + \left(\Pi^R\left(\bar z_2, \psi'(\bar z_2)\right) 1_{z\in\Omega^R} + \Pi^L\left(\bar z_2, \psi'(\bar z_2)\right) 1_{z\in\Omega^L}\right) z_1. \tag{7.2}$$

The following theorem is reminiscent of [11, Theorem 2.6].

Theorem 7.2. *Let* $u : \mathbb{R}^2 \to \mathbb{R}$ *be a subsolution (resp. supersolution) of* (1.19) *and a map* $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$, $(z_2,p_2) \mapsto \big(\Pi^L(z_2,p_2), \Pi^R(z_2,p_2)\big)$ *such that* (7.1) *holds for all* $(z_2,p_2) \in \mathbb{R}^2$. *The function* u *is a subsolution (resp. supersolution) of* (1.20) *if and only if for any* $z = (0, z_2) \in \Gamma$ *and for all* $\varphi \in \mathcal{R}^\Pi(z)$ *such that* $u - \varphi$ *has a local maximum (resp. local minimum) at* z,

$$\lambda u(z) + \max\left(E(z_2, \partial_{z_2}\varphi(z)), H_\Gamma(z, D\varphi^L(z), D\varphi^R(z))\right) \le 0, \quad (\textit{resp.} \ \ge 0), \tag{7.3}$$

where the meaning of $D\varphi^L$ *and* $D\varphi^R$ *is given in Definition 1.2.*

Proof. The proof follows the lines of that of [11, Theorem 2.6] and is given in Appendix B for the reader's convenience. It is worth to note that Lemma 3.2 is important in order to use the arguments contained in the proof of [11, Theorem 2.6]. □

7.2 Proof of Theorem 1.5

As seen in § 3, the result will be proved if we show that the sequence $(\tilde v_\varepsilon)_\varepsilon$ corresponding to the straightened geometry converges to v. We will actually prove that $\overline{\tilde v}$ and $\underline{\tilde v}$ defined in (3.1) are respectively a subsolution and a supersolution of (1.21). From Theorem 6.4, this will imply that $\overline{\tilde v} = \underline{\tilde v} = v = \lim_{\varepsilon\to 0} \tilde v_\varepsilon$. Moreover, from Proposition 3.1, we just have to check the transmission condition (1.20).

We restrict ourselves to checking that $\overline{\tilde v}$ is a subsolution of (1.21), since the proof that $\underline{\tilde v}$ is a supersolution of (1.21) is similar. Take $\bar z = (0, \bar z_2) \in \Gamma$. We are going to use Theorem 7.2 with the special choice for the map $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$: $\Pi(z_2,p_2) = \left(\overline{\Pi}^L(z_2,p_2), \overline{\Pi}^R(z_2,p_2)\right)$. Take a test-function $\varphi \in \mathcal{R}^\Pi(\bar z)$, i.e. of the form

$$\varphi(z) = \psi(z_2) + \left(\overline{\Pi}^R(\bar z_2, \psi'(\bar z_2)) 1_{z\in\Omega^R} + \overline{\Pi}^L(\bar z_2, \psi'(\bar z_2)) 1_{z\in\Omega^L}\right) z_1, \tag{7.4}$$

for a $\mathcal{C}^1$ function $\psi : \mathbb{R} \to \mathbb{R}$, such that $\overline{\tilde v} - \varphi$ has a strict local maximum at $\bar z$ and that $\overline{\tilde v}(\bar z) = \varphi(\bar z)$.
Let us proceed by contradiction and assume that

$$\lambda\varphi(\bar z) + \max\left(E(\bar z_2, \partial_{z_2}\varphi(\bar z)), H_\Gamma(\bar z, D\varphi^L(\bar z), D\varphi^R(\bar z))\right) = \theta > 0. \tag{7.5}$$

From (7.4), we see that $H_\Gamma(\bar z, D\varphi^L(\bar z), D\varphi^R(\bar z)) \le E(\bar z_2, \partial_{z_2}\varphi(\bar z))$ and (7.5) is equivalent to

$$\lambda\psi(\bar z_2) + E(\bar z_2, \psi'(\bar z_2)) = \theta > 0. \tag{7.6}$$

Let $\chi(\bar z_2, \psi'(\bar z_2), \cdot)$ be a solution of (4.19) such that $\chi(\bar z_2, \psi'(\bar z_2), (0,0)) = 0$, see Theorem 4.9, and $W(\bar z_2, \psi'(\bar z_2), z_1) = \lim_{\varepsilon\to 0} \varepsilon\chi(\bar z_2, \psi'(\bar z_2), \frac{z}{\varepsilon})$.

Step 1 We claim that for $\varepsilon > 0$ and $r > 0$ small enough, the function φ^ε:

$$\varphi^\varepsilon(z) = \psi(z_2) + \varepsilon\chi(\bar z_2, \psi'(\bar z_2), \frac{z}{\varepsilon})$$

is a viscosity supersolution of

$$\begin{cases} \lambda\varphi^\varepsilon(z) + \tilde H^i_\varepsilon(z, D\varphi^\varepsilon(z)) & \geq \frac{\theta}{2} \qquad \text{if } z \in \Omega^i \cap B((0,\bar z_2), r), \\ \lambda\varphi^\varepsilon(z) + \tilde H_{\Gamma,\varepsilon}(z, D\left(\varphi^\varepsilon\right)^L(z), D\left(\varphi^\varepsilon\right)^R(z)) & \geq \frac{\theta}{2} \qquad \text{if } z \in \Gamma \cap B((0,\bar z_2), r), \end{cases} \tag{7.7}$$

where the Hamiltonians $\tilde H^i_\varepsilon$ and $\tilde H_{\Gamma,\varepsilon}$ are defined by (2.11) and (2.12).
Indeed, if ξ is a test-function in $\mathcal{R}$ such that $\varphi^\varepsilon - \xi$ has a local minimum at $z^\star \in B((0,\bar z_2), r)$, then, from the definition of φ^ε, $y \mapsto \chi(\bar z_2, \psi'(\bar z_2), y) - \frac{1}{\varepsilon}\left(\xi(\varepsilon y) - \psi(\varepsilon y_2)\right)$ has a local minimum at $\frac{z^\star}{\varepsilon}$. Let us now use the fact that $y \mapsto \chi(\bar z_2, \psi'(\bar z_2), y)$ is a supersolution of (4.19):

If $\frac{z^\star}{\varepsilon} \in \Omega^i$, for $i = L$ or R, then $\tilde H^i((0,\bar z_2), D\xi(z^\star) - \psi'(z_2^\star)e_2 + \psi'(\bar z_2)e_2, \frac{z_2^\star}{\varepsilon}) \geq E(\bar z_2, \psi'(\bar z_2))$. From the regularity properties of $\tilde H^i$, see Lemma 4.1,

$$\tilde H^i((0,\bar z_2), D\xi(z^\star) - \psi'(z_2^\star)e_2 + \psi'(\bar z_2)e_2, \frac{z_2^\star}{\varepsilon}) = \tilde H^i_\varepsilon(z^\star, D\xi(z^\star)) + o_{\varepsilon\to 0}(1) + o_{r\to 0}(1),$$

thus

$$\begin{aligned} &\lambda\varphi^\varepsilon(z^\star) + \tilde H^i_\varepsilon(z^\star, D\xi(z^\star)) \\ \geq& E(\bar z_2, \psi'(\bar z_2)) + \lambda\left(\psi(z_2^\star) + \varepsilon\chi\left(\bar z_2, \psi'(\bar z_2), \frac{z^\star}{\varepsilon}\right)\right) + o_{\varepsilon\to 0}(1) + o_{r\to 0}(1). \end{aligned}$$

From (7.6), this implies that

$$\lambda\varphi^\varepsilon(z^\star) + \tilde H^i_\varepsilon(z^\star, D\xi(z^\star)) \geq \theta + \lambda\varepsilon\chi\left(\bar z_2, \psi'(\bar z_2), \frac{z^\star}{\varepsilon}\right) + o_{\varepsilon\to 0}(1) + o_{r\to 0}(1).$$

Recall that the function $y \mapsto \varepsilon\chi(\bar z_2, \psi'(\bar z_2), \frac{y}{\varepsilon})$ converges locally uniformly to $y \mapsto W(\bar z_2, \psi'(\bar z_2), y)$, which is a Lipschitz continuous function, independent of y_2 and such that $W(\bar z_2, \psi'(\bar z_2), 0) = 0$. Therefore, for ε and r small enough, $\lambda\varphi^\varepsilon(z^\star) + \tilde H^i_\varepsilon(z^\star, D\xi(z^\star)) \geq \frac{\theta}{2}$.

If $\frac{z^\star}{\varepsilon} \in \Gamma$, then, for some $i \in \{L, R\}$, $\tilde H^{+,i}(\bar z_2, D\xi^i(z^\star) - \psi'(z_2^\star)e_2 + \psi'(\bar z_2)e_2, \frac{z_2^\star}{\varepsilon}) \geq E(\bar z_2, \psi'(\bar z_2))$. Since the Hamiltonian $\tilde H^{+,i}$ enjoys the same regularity properties as $\tilde H^i$, see Lemma 4.1, it is possible to use the same arguments as in the case $\frac{z^\star}{\varepsilon} \in \Omega^i$. For r and ε small enough, $\lambda\varphi^\varepsilon(z^\star) + \tilde H^{+,i}_{\Gamma,\varepsilon}(z^\star, D\xi(z^\star)) \geq \frac{\theta}{2}$. The claim that φ^ε is a supersolution of (7.7) is proved.

Step 2 Let us prove that there exist some positive constants $K_r > 0$ and $\varepsilon_0 > 0$ such that

$$\tilde v^\varepsilon(z) + K_r \leq \varphi^\varepsilon(z), \quad \forall z \in \partial B(\bar z, r),\ \forall \varepsilon \in (0, \varepsilon_0). \tag{7.8}$$

Indeed, since $\overline{\tilde v} - \varphi$ has a strict local maximum at $\bar z$ and since $\overline{\tilde v}(\bar z) = \varphi(\bar z)$, there exists a positive constant $\tilde K_r > 0$ such that $\overline{\tilde v}(z) + \tilde K_r \leq \varphi(z)$ for any $z \in \partial B(\bar z, r)$. Since $\overline{\tilde v} = \limsup_\varepsilon{}^* \tilde v_\varepsilon$, there exists $\tilde\varepsilon_0 > 0$ such that $\tilde v^\varepsilon(z) + \frac{\tilde K_r}{2} \leq \varphi(z)$ for any $0 < \varepsilon < \tilde\varepsilon_0$ and $z \in \partial B(\bar z, r)$. But $z \mapsto \varphi^\varepsilon(z)$ converges locally uniformly to $z \mapsto \psi(z_2) + W(\bar z_2, \psi'(\bar z_2), z)$ as ε tends to 0. Hence, thanks to (5.14) in Corollary 5.6,

$$\psi(z_2) + W(\bar z_2, \psi'(\bar z_2), z) \geq \psi(z_2) + \left(\overline{\Pi}^R(\bar z_2, \psi'(\bar z_2))1_{z\in\Omega^R} + \overline{\Pi}^L(\bar z_2, \psi'(\bar z_2))1_{z\in\Omega^L}\right) z_1,$$

and we get (7.8) for some constants $K_r > 0$ and $\varepsilon_0 > 0$.

Step 3 From the previous steps and the local comparison principle in Theorem 2.5, we find that for r and ε small enough,

$$\tilde{v}^\varepsilon(z) + K_r \le \varphi^\varepsilon(z) \qquad \forall z \in B(\bar{z}, r).$$

Taking $z = \bar{z}$ and letting $\varepsilon \to 0$, we obtain

$$\overline{\tilde{v}}(\bar{z}) + K_r \le \psi(\bar{z}_2) = \varphi(\bar{z}) = \overline{\tilde{v}}(\bar{z}),$$

which cannot happen. The proof is completed.

Remark 7.3. *For the proof of the supersolution inequality, the test-function φ should be chosen of the form*

$$\varphi(z_1, z_2) = \psi(z_2) + 1_{\Omega^R}\widehat{\Pi}^R(\bar{z}_2, \psi'(\bar{z}_2))z_1 + 1_{\Omega^L}\widehat{\Pi}^L(\bar{z}_2, \psi'(\bar{z}_2))z_1,$$

where $\psi \in \mathcal{C}^1(\mathbb{R})$ and for $i = L, R$, $\widehat{\Pi}^i(\bar{z}_2, \psi'(\bar{z}_2))$ are defined in (4.27) *and* (4.29).

A Proof of Lemma 4.4

Subsolutions Let $\varphi \in \mathcal{R}_\rho$ be a test-function and $\bar{y} \in [-\rho, \rho] \times \mathbb{R}$ be such that $u^0 - \varphi$ has a strict local maximum at $\bar{y}$. If $\bar{y} \in (-\rho, 0) \times \mathbb{R}$, (resp. $\bar{y} \in (0, \rho) \times \mathbb{R}$) is standard to check that $\tilde{H}^L((0, z_2), D(\bar{y}) + p_2e_2, \bar{y}_2) \le \lambda$, (resp. $\tilde{H}^R((0, z_2), D(\bar{y}) + p_2e_2, \bar{y}_2) \le \lambda$). We may focus on the case when $\bar{y} = (0, \bar{y}_2) \in \Gamma$, because the cases $\bar{y}_1 = \pm\rho$ can be treated with similar but simpler arguments. We wish to prove that

$$\tilde{H}^{+,i}((0, z_2), D\varphi^i(\bar{y}) + p_2e_2\bar{y}_2) \le \lambda, \quad \forall i = L, R. \tag{A.1}$$

We may assume that for all $\eta \in [0, 1]$, the function $u^\eta - \varphi$ is Lipschitz continuous in $[-\rho, \rho] \times \mathbb{R}$ with a Lipschitz constant $\bar{L}$ independent of η. Fix $i = L, R$, we define the distance d_i to $\bar{\Omega}_i$ by

$$d_i(y) = \begin{cases} 0 & \text{if } y \in \Omega^i, \\ |y_1| & \text{otherwise.} \end{cases}$$

Clearly, $d_i \in \mathcal{R}_\rho$. Take $C = \bar{L} + 1$. The function $y \mapsto u^0(y) - \varphi(y) - Cd_i(y)$ has a strict local maximum at $\bar{y}$. Thanks to the local uniform convergence of u^η to u^0, there exists $r \in (0, \rho)$ and a sequence of points $y^\eta \in B(\bar{y}, r)$ such that

$$u^\eta(y) - \varphi(y) - Cd_i(y) \le u^\eta(y^\eta) - \varphi(y^\eta) - Cd_i(y^\eta), \qquad \text{for all } y \in B(\bar{y}, r).$$

Up to the extraction of a subsequence, we may assume that $y^\eta \to \bar{y}$ as η tends to 0. Note that $y^\eta \in \bar{\Omega}^i$. Indeed, if it was not the case, then calling $\overline{y^\eta} = (0, y_2^\eta) \in B(\bar{y}, r)$,

$$u^\eta(y^\eta) - \varphi(y^\eta) - (u^\eta(\overline{y^\eta}) - \varphi(\overline{y^\eta})) \le \bar{L}|y_1^\eta| = \bar{L}d_i(y^\eta),$$

and $u^\eta(y^\eta) - \varphi(y^\eta) - Cd_i(y^\eta) \le u^\eta(\overline{y^\eta}) - \varphi(\overline{y^\eta}) - d_i(y^\eta) < u^\eta(\overline{y^\eta}) - \varphi(\overline{y^\eta}) - Cd_i(\overline{y^\eta})$, in contradiction with the definition of y^η.
Up to the extraction of subsequences, we can make out two cases:
Case 1: $y^\eta \in \Gamma$. We obtain $\eta u^\eta(y^\eta) + \max_{i=L,R}\left(\tilde{H}^{+,i}((0, z_2), Du^i(y^\eta) + p_2e_2, y_2^\eta)\right) \le \lambda_\eta$, and then (A.1) by letting $\eta \to 0$.
Case 2: $y^\eta \in \Omega^i$. We obtain that $\eta u^\eta(y^\eta) + \tilde{H}^i((0, z_2), D\varphi(y^\eta) + p_2e_2, y_2^\eta) \le \lambda_\eta$, and then by letting $\eta \to 0$ that $\tilde{H}^i((0, z_2), D\varphi(\bar{y}) + p_2e_2, \bar{y}_2) \le \lambda$, from the continuity of the Hamiltonian $\tilde{H}^i$. Finally, since $\tilde{H}^i - \tilde{H}^{+,i} \ge 0$, which yields that $\tilde{H}^{+,i}((0, z_2), D\varphi^i(\bar{y}) + p_2e_2\bar{y}_2) \le \lambda$.
Since the arguments above can be applied for $i = L$ and $i = R$, we have obtained (A.1).

Supersolutions Let $\varphi \in \mathcal{R}_\rho$ be a test-function and $\bar{y} \in [-\rho, \rho] \times \mathbb{R}$ be such that $u^0 - \varphi$ has a strict local minimum at $\bar{y}$. As above, we may focus on the case when $\bar{y} \in \Gamma$. We wish to prove that

$$\max_{i=L,R} \left(\tilde{H}^{+,i}((0, z_2), D\varphi^i(\bar{y}) + p_2 e_2, \bar{y}_2) \right) \geq \lambda. \tag{A.2}$$

Define

$$\begin{aligned} \tilde{p}_0^L &= \\ \min & \left\{ \begin{array}{l} p \in \mathbb{R} : \\ \tilde{H}^L((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) = \tilde{H}^{+,L}((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) \end{array} \right\}, \\ \tilde{p}_0^R &= \\ \max & \left\{ \begin{array}{l} p \in \mathbb{R} : \\ \tilde{H}^R((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) = \tilde{H}^{+,R}((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) \end{array} \right\} \end{aligned}$$

Recall that thanks to Lemma 3.2 and Remark 4.3,

$$\begin{aligned} \tilde{H}^{+,L}((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) &= \left\{ \begin{array}{ll} \tilde{E}_0^L(z_2, \partial_{y_2}\varphi(\bar{y}) + p_2, \bar{y}_2) & \text{if } p \leq \tilde{p}_0^L, \\ \tilde{H}^1((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) & \text{if } p \geq \tilde{p}_0^L, \end{array} \right. \\ \tilde{H}^{+,R}((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) &= \left\{ \begin{array}{ll} \tilde{H}^2((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2) & \text{if } p \leq \tilde{p}_0^R, \\ \tilde{E}_0^R(z_2, \partial_{y_2}\varphi(\bar{y}) + p_2, \bar{y}_2) & \text{if } p \geq \tilde{p}_0^R. \end{array} \right. \end{aligned}$$

where $\tilde{E}_0^i(z_2, p_2, y_2)$ is defined in Remark 4.3.
We make out two cases:
Case 1: $\partial_{y_1}\varphi^L(\bar{y}) \geq \tilde{p}_0^L$, $\partial_{y_1}\varphi^R(\bar{y}) \leq \tilde{p}_0^R$ and for each $i \in \{L, R\}$

$$\tilde{H}^i((0, z_2), D\varphi^i(\bar{y}) + p_2 e_2, \bar{y}_2) = \max_{j=L,R} \left(\tilde{H}^{+,j}((0, z_2), D\varphi^j(\bar{y}) + p_2 e_2, \bar{y}_2) \right). \tag{A.3}$$

In this case, we can use a standard stability argument: there exists a sequence y^η of local minimum points of $u^\eta - \varphi$ which converges to $\bar{y}$ as η tends to 0. If, for a subsequence still called y^η, $y^\eta \in \Gamma$, then $\eta u^\eta(y^\eta) + \max_{i=L,R} \left(\tilde{H}^{+,i}((0, z_2), D\varphi^i(y^\eta) + p_2 e_2, y_2^\eta) \right) \geq \lambda_\eta$, because u^η is a supersolution of (4.9), and (A.2) is obtained by letting $\eta \to 0$.
If for a subsequence, $y^\eta \in \Omega^i$ for some $i \in \{L, R\}$, then $\eta u^\eta(y^\eta) + \tilde{H}^i((0, z_2), D\varphi(y^\eta) + p_2 e_2, y_2^\eta) \geq \lambda_\eta$, and by letting $\eta \to 0$, $\tilde{H}^i((0, z_2), D\varphi^i(\bar{y}) + p_2 e_2, \bar{y}_2) \geq \lambda$. Finally, (A.2) is obtained thanks to (A.3).
Case 2: the assumptions of the case 1 are not satisfied. Thanks to the above identities for $\tilde{H}^{+,R}((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2)$ and $\tilde{H}^{+,L}((0, z_2), (\partial_{y_2}\varphi(\bar{y}) + p_2)e_2 + pe_1, \bar{y}_2)$, by modifying the slopes of φ in the normal direction on each side of Γ, it is possible to construct a test-function $\psi \in \mathcal{R}_\rho$ such that $\psi(\bar{y}) = \varphi(\bar{y})$, $\partial_{y_2}\psi(\bar{y}) = \partial_{y_2}\varphi(\bar{y})$, $\partial_{y_1}\psi^L(\bar{y}) \geq \partial_{y_1}\varphi^L(\bar{y})$, $\partial_{y_1}\psi^R(\bar{y}) \leq \partial_{y_1}\varphi^R(\bar{y})$, and ψ satisfies (A.3) for each $i \in \{L, R\}$. Thus, since ψ touches φ at $\bar{y}$ from below, $\bar{y}$ is still a strict local minimum point of $u^0 - \psi$ and we conclude by applying the result proved in the first case.

B Proof of Theorem 7.2

The proof of Theorem 7.2 follows the lines of [11, Theorem 2.6] and relies on the following two technical lemmas, which can be proved by adapting the proofs of Lemma 2.8 and Lemma 2.9 in [11], with very slight changes.

Lemma B.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a subsolution of* (1.19) *and $\phi \in \mathcal{R}$ touching u from above at $\bar{z} = (0, \bar{z}_2) \in \Gamma$. For each $i \in \{L, R\}$, the real number $\bar{p}_i$:*

$$\bar{p}_i = \inf \left\{ p \in \mathbb{R} \ : \ \exists r > 0 \ s.t. \ \phi(z) + \sigma^i p z_1 \geq u(z), \ \forall z = (z_1, z_2) \in [0, r) \times (\bar{z}_2 - r, \bar{z}_2 + r) \right\},$$

where σ^i is given by (1.4)*, is nonpositive. Moreover,*

$$\lambda u(0, \bar{z}_2) + H^i((0, \bar{z}_2), D\phi^i(0, \bar{z}_2) + \sigma^i \bar{p}_i e_1) \leq 0. \tag{B.1}$$

Lemma B.2. *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a supersolution of* (1.19) *and $\phi \in \mathcal{R}$ touching w from below at $\bar{z} = (0, \bar{z}_2) \in \Gamma$. For each $i \in \{L, R\}$. For each $i \in \{L, R\}$, the real number $\tilde{p}_i$:*

$$\tilde{p}_i = \sup \left\{ p \in \mathbb{R} \ : \ \exists r > 0 \ s.t. \ \phi(z) + \sigma^i p z_1 \leq w(z), \ \forall z = (z_1, z_2) \in [0, r) \times (\bar{z}_2 - r, \bar{z}_2 + r) \right\},$$

is nonnegative. Moreover

$$\lambda w(0, \bar{z}_2) + H^i((0, \bar{z}_2), D\phi^i(0, \bar{z}_2) + \sigma^i \tilde{p}_i e_1) \geq 0. \tag{B.2}$$

We are now ready to prove Theorem 7.2.

Subsolutions Let Π be a map as in Definition 7.1. Suppose that a subsolution u of (1.19) satisfies (7.3) for all $z \in \Gamma$ and all test-functions in $\mathcal{R}^{\Pi}(z)$ touching u from above at z.
Let $\phi \in \mathcal{R}$ be such that $u - \phi$ has a strict local maximum at $\bar{z} \in \Gamma$ and that $u(\bar{z}) = \phi(\bar{z})$. We wish to prove that

$$\lambda u(\bar{z}) + \max \left(E(\bar{z}_2, \partial_{z_2} \phi(\bar{z})), H_\Gamma(z, D\phi^L(\bar{z}), D\phi^R(\bar{z})) \right) \leq 0. \tag{B.3}$$

From Lemma B.1, for each $i \in \{L, R\}$, there exists $\bar{p}_i \leq 0$ such that

$$\lambda u(\bar{z}) + H^i(\bar{z}, D\phi^i(\bar{z}) + \sigma^i \bar{p}_i e_1) \leq 0. \tag{B.4}$$

From the monotonicity properties of the Hamiltonians $H^{+,i}$ stated in Lemma 3.2,

$$\begin{aligned} H_\Gamma(z, D\phi^L(\bar{z}), D\phi^R(\bar{z})) &\leq H_\Gamma(z, D\phi^L(\bar{z}) - \bar{p}_L e_1, D\phi^R(\bar{z}) + \bar{p}_R e_1) \\ &\leq \max \left(H^L(z, D\phi^L(\bar{z}) - \bar{p}_L e_1), H^R(z, D\phi^R(\bar{z}) + \bar{p}_R e_1) \right). \end{aligned}$$

Hence, from (B.4),

$$\lambda u(\bar{z}) + H_\Gamma(\bar{z}, D\phi^L(\bar{z}), D\phi^R(\bar{z})) \leq 0.$$

Therefore, in order to prove (B.3), we are left with checking that

$$\lambda u(\bar{z}) + E(\bar{z}_2, \partial_{z_2} \phi(\bar{z})) \leq 0. \tag{B.5}$$

Recall that from Proposition 4.11, $E(\cdot, \cdot) \geq E_0(\cdot, \cdot)$. If $E(\bar{z}_2, \partial_{z_2} \phi(\bar{z})) = E_0(\bar{z}_2, \partial_{z_2} \phi(\bar{z}))$, then (B.5) is a direct consequence of (B.4). Let us consider the case when $E(\bar{z}_2, \partial_{z_2} \phi(\bar{z})) > E_0(\bar{z}_2, \partial_{z_2} \phi(\bar{z}))$ and assume by contradiction that

$$\lambda u(\bar{z}) + E(\bar{z}_2, \partial_{z_2} \phi(\bar{z})) > 0. \tag{B.6}$$

Then, from (B.4) , for any $i \in \{L, R\}$,

$$H^{-,i}(\bar{z}, D\phi^i(\bar{z}) + \sigma^i \bar{p}_i e_1) \leq -\lambda u(\bar{z}) < E(\bar{z}_2, \partial_{z_2} \phi(\bar{z})).$$

From this and the monotonicity properties of the functions $p \in \mathbb{R} \mapsto H^{-,i}(z, \partial_{z_2}\phi(\bar{z})e_2 + pe_1)$, we deduce that

$$\Pi^L(\bar{z}_2, \partial_{z_2}\phi(\bar{z})) < \partial_{z_1}\phi^L(\bar{z}) - \bar{p}_L, \quad \text{and} \quad \Pi^R(\bar{z}_2, \partial_{z_2}\phi(\bar{z})) > \partial_{z_1}\phi^R(\bar{z}) + \bar{p}_R.$$

Thus, the modified test-function $\varphi \in \mathcal{R}^\Pi(\bar{z})$ defined by

$$\varphi(z) = \phi(0, z_2) + 1_{\Omega^L}(z)\Pi^L(\bar{z}_2, \partial_{z_2}\phi(\bar{z}))z_1 + 1_{\Omega^R}(z)\Pi^R(\bar{z}_2, \partial_{z_2}\phi(\bar{z}))z_1$$

is such that $u - \varphi$ has a local maximum at $\bar{z}$, and therefore

$$\lambda u(\bar{z}) + \max\left(E(\bar{z}_2, \partial_{z_2}\varphi(\bar{z})), H_\Gamma(\bar{z}, D\varphi^L(\bar{z}), D\varphi^R(\bar{z}))\right) \leq 0,$$

which contradicts (B.6).

Supersolutions Suppose that a supersolution u of (1.19) satisfies (7.3) for all $z \in \Gamma$ and all test-functions in $\mathcal{R}^\Pi(z)$ touching u from below at z.
Let $\phi \in \mathcal{R}$ be such that $u - \phi$ has a strict local maximum at $\bar{z} \in \Gamma$ with $u(\bar{z}) = \phi(\bar{z})$. We wish to prove that

$$\lambda u(\bar{z}) + \max\left(E(\bar{z}_2, \partial_{z_2}\phi(\bar{z})), H_\Gamma(z, D\phi^L(\bar{z}), D\phi^R(\bar{z}))\right) \geq 0. \tag{B.7}$$

From Lemma B.2, for each $i \in \{L, R\}$ there exists $\tilde{p}_i \geq 0$ such that

$$\lambda u(\bar{z}) + H^i(\bar{z}, D\phi^i(\bar{z}) + \sigma^i\tilde{p}_i e_1) \geq 0, \tag{B.8}$$

and using the monotonicity properties of the Hamiltonians $H^{+,i}$, see Lemma 3.2,

$$H_\Gamma(\bar{z}, D\phi^L(\bar{z}), D\phi^R(\bar{z})) \geq H_\Gamma(\bar{z}, D\phi^L(\bar{z}) - \tilde{p}_L e_1, D\phi^R(\bar{z}) + \tilde{p}_R e_1). \tag{B.9}$$

If for some $i \in \{L, R\}$, $H^{+,i}(\bar{z}, D\phi^i(\bar{z}) + \sigma^i\tilde{p}_i e_1) = H^i(\bar{z}, D\phi^i(\bar{z}) + \sigma^i\tilde{p}_i e_1)$, then (B.7) follows readily from (B.8) and (B.9) so we may now suppose that

$$H^i(\bar{z}, D\phi^i(\bar{z}) + \sigma^i\tilde{p}_i e_1) = H^{-,i}(\bar{z}, D\phi^i(\bar{z}) + \sigma^i\tilde{p}_i e_1), \quad i = L, R. \tag{B.10}$$

Assume by contradiction that $\lambda u(\bar{z}) + \max\left(E(\bar{z}_2, \partial_{z_2}\phi(\bar{z})), H_\Gamma(\bar{z}, D\phi^L(\bar{z}), D\phi^R(\bar{z}))\right) < 0$. Thus, from (B.8) and (B.10),

$$E(\bar{z}_2, \partial_{z_2}\phi(\bar{z})) < -\lambda u(\bar{z}) \leq H^{-,i}(\bar{z}, D\phi^i(\bar{z}) + \sigma^i\tilde{p}_i e_1), \quad i = L, R. \tag{B.11}$$

From (B.10), (B.11) and the monotonicity properties of the functions $p \in \mathbb{R} \mapsto H^{-,i}(z, \partial_{z_2}\phi(\bar{z})e_2 + pe_1)$, we deduce that

$$\Pi^L(\bar{z}_2, \partial_{z_2}\phi(\bar{z})) > \partial_{z_1}\phi^L(\bar{z}) - \tilde{p}_L, \quad \text{and} \quad \Pi^R(\bar{z}_2, \partial_{z_2}\phi(\bar{z})) < \partial_{z_1}\phi^R(\bar{z}) + \tilde{p}_R.$$

Since the modified test-function $\varphi \in \mathcal{R}^\Pi(\bar{z})$,

$$\varphi(z) = \phi(0, z_2) + 1_{\Omega^L}\Pi^L(\bar{z}_2, \partial_{z_2}\phi(\bar{z}))z_1 + 1_{\Omega^R}\Pi^R(\bar{z}_2, \partial_{z_2}\phi(\bar{z}))z_1,$$

is such that $u - \varphi$ has a local minimum at $\bar{z}$, we get

$$\lambda u(\bar{z}) + \max\left(E(\bar{z}_2, \partial_{z_2}\phi(\bar{z})), H_\Gamma(z, D\varphi^L(\bar{z}), D\varphi^R(\bar{z}))\right) \geq 0,$$

which is the desired contradiction.

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